Decentralized Matching: The Role of Commitment

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Abstract

Gale and Shapley's matching model has a "stable matching," where no pair of a firm and a worker prefer each other. The main application has been a one-time matching with a centralized matchmaking mechanism. However, not many job markets are centralized, and participants may remain active in the market even after they are matched. This paper studies an infinite-horizon decentralized game. Every period, firms with vacant positions make offers to workers, who then decide which offers to accept. The game depends on whether agents commit to their relationships. With no commitment, a worker can leave the current employer but may also be dismissed. With two-sided commitment, matched pairs withdraw from the market. With one-sided commitment, workers are protected from dismissal but remain active in the market, as in the case of tenured professors. We characterize stationary equilibria for each commitment structure. Without commitment, equilibrium outcomes coincide with stable matchings; neither side of the market is favored and the set of unemployed workers is equilibrium-invariant. With commitment, either one-sided or two-sided, an equilibrium may reach an unstable matching even if there is no initial commitment. With one-sided commitment, an equilibrium may even yield an unstable matching where all workers are worse off than in every stable matching. In this case, the workers are better off if job protections are removed.

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1 Introduction

The basic idea of matching theory is that a matching between agents is not stable if there exists a pair of agents who prefer each other to their current partners. Such a pair is called a "blocking pair," and matchings with no blocking pair are called "stable matchings." For the standard matching problem, a stable matching exists for any number of agents and any profile of preferences, as shown by Gale and Shapley (1962). Gale and Shapley also give an algorithm that finds a stable matching. The algorithm has been used since early 1950s by a centralized matchmaking mechanism to assign new American physicians to hospitals (Roth, 1984). The hypothesis of Roth (1991) is that the success of a centralized labor market depends on whether the matchmaking mechanism generates a stable matching.

The focus of the literature has been matching markets that are centralized, where participants submit preferences to a center and a matching is determined by an algorithm. On the other hand, many markets—including the market for economists—are decentralized, where a matching is determined by a series of offers and replies chosen strategically by the agents. How does a decentralized market compare with a centralized one? Does a decentralized market generate a stable matching? Is the answer sensitive to the way in which the decentralized market operates?

To address these questions, this paper studies equilibria in a dynamic game of matching. We extend the original Gale–Shapley model to a dynamic and noncooperative setting where firms and workers interact repeatedly in a decentralized manner. In every period, firms with vacant positions make offers to workers, who then choose individually which offer to accept. To raise realism, we assume that each worker observes only the offers made to her. The market takes place every period—for example, once a year—and all agents derive utility from their matching in each period. We focus on stationary equilibria, where actions vary only with the payoff-relevant state of the game.

Our dynamic matching model makes it possible to analyze the role of commitment, in particular, how equilibrium matching depends on whether agents commit themselves to their employment relationships. In our stylized model, three interesting possibilities can be considered. The first possibility is that, once a pair of agents are matched, they withdraw from the market and stay together permanently. The second possibility is that agents make no commitment beyond one period. That is, employees can accept new job offers but may also be dismissed. The third possibility is that firms make commitments while workers do not. That is, workers remain active in the job market but are protected from dismissal, as in the case of tenured professors and government employees.

In the absence of commitment, we show that every stationary equilibrium yields a stable matching. An implication is that if a worker is unemployed in one equilibrium, he is unemployed in every equilibrium. We also show that, in the absence of commitment, stationary equilibria can yield *any* stable matching: no stable matchings are ruled out. This is in contrast with an important result in the matching theory: Gale–Shapley algorithm favors one side of the market. In the static model, if one uses Gale–Shapley algorithm where firms make offers and workers passively respond to the offers, the algorithm generates what is known as the firm-optimal stable matching, which is a stable matching that is unambiguously best for all firms and worst for all workers. If the roles of firms and workers are reversed in the algorithm, the outcome changes to the worker-optimal stable matching. The result has an obvious practical importance: the American hospital-intern matching market used the hospital-optimal algorithm for 50 years but recently changed the algorithm to the student-optimal one (Roth and Peranson, 1999). In contrast, the result in the present paper shows that, in decentralized and dynamic matching markets with no commitment, which side of the market makes offers is irrelevant.

On the other hand, if at least one side of the market makes commitment, a stationary equilibrium need not yield a stable matching. Of course, the result is trivial if the game starts with many committed firms and workers and no stable matching is reachable. However, the result obtains even if agents start with no commitment. Without any initial commitment and without any random shock or mistake, it is possible that agents knowingly and willingly reach an unstable matching and stay there. A blocking pair exists in the equilibrium matching but does not get together. For the one-sided commitment case, the reason is that, if the firm in the blocking pair makes an offer to the blocking partner, the action triggers a chain reaction in which a firm that lost a worker takes another worker from another firm. In the end, the firm who started the process ends up losing the blocking partner. Anticipating this, the firm in the blocking pair does not make an offer to the blocking partner.

For the one-sided commitment case, it is even possible that the equilibrium outcome is an unstable matching where all workers are worse off than in any stable matching. This is interesting since job protections are intended to protect workers. As the model predicts, if job protections are lifted, every equilibrium yields a stable matching. Therefore, in the example, all workers are better off if job protections are removed.

An example of workers who are protected from dismissal is public school teachers in the US. In most states, teachers in public schools are strongly protected by state law and the collective bargaining agreement between the district and the teachers' union. Dismissing a teacher is known to be difficult and hence rare because the principal has to go through a lengthy legal procedure (Bridges, 1992; Ballou, 1999). Weakening job protections for public school teachers has been a major political issue but met a strong resistance from teachers' unions. For example, in 2005, Governor Schwarzenegger of California proposed a measure, called Proposition 74, to make it easier to dismiss teachers. According to the press, California Teachers Association spent more than $7 million to defeat the measure. The measure was indeed rejected by the public in a special election.

The results of this study suggest that job protections need not be good for teachers. The result for the no-commitment case shows that, without job protections, the market reaches a stable matching and has no systematic bias towards either side of the market. With job protections, unstable matchings are possible but teachers may not gain from it. It is theoretically possible that job protections make all teachers worse off.

There are a few papers that also study decentralized matching markets. Haeringer and Wooders (2006) study a similar game but there are a few critical differences. They assume that once a pair forms a partnership, it is the end of the game for the pair, as in our two-sided commitment case. Furthermore, they assume that the payoff realizes only once and depends only on the identity of the partner. There are no time preferences. Their model therefore can be thought of as describing a job market for a single year. Our model, on the other hand, describes the dynamics of a job market over years where agents collect payoffs as they go along. Technically, their model is similar to a bargaining model while ours is similar to a repeated game.

Konishi and Sapozhnikov (2005) and Niederle and Yariv (2007) also consider similar models with realistic details on salary or the length of offers. But, as in Haeringer and Wooders, agents who get matched exit the game.

There is also a literature of search models of matching. Our model is not a search model: we assume that market participants know each other well and do not have to rely on random encounters. Within the search-model literature, Adachi (2003) is particularly close to our paper since it is also based on the Gale–Shapley model. But, as in the above papers, he also deals only with the case where agents exit as soon as they get matched. Further, to make the distribution of agents stationary, he uses a "replacement assumption": when agents exit, their clones take their places. He shows that, as the discount factor goes to one, the set of equilibrium outcomes converges to the set of stable matchings. As we show, the result does not hold if agents know each other and no replacement arrives, even if the discount factor is close to one.

Blum, Roth, and Rothblum (1997) consider an algorithm that finds a stable matching when some of the agents are initially matched. Towards the end of the paper, the authors study a game similar to ours with one-sided commitment. However, the payoff realizes only once and depends only on the final matching. The paper characterizes Nash equilibria in "preference strategies," where each agent uses a single (possibly false) preference ordering for decisions at all nodes [see also Pais (2005) for an extension].

Alcalde and Romero-Medina (2000), in the context of mechanism design theory, study a game where only one round of offers and replies takes place. That is, firms make offers, workers reply, and the game ends. They show that this game achieves (or implements) stable matchings [see also Alcalde, Pérez-Castrillo, and Romero-Medina (1998)].

There is also a strand of literature that studies, within the original static model, whether a myopic adjustment process based on the blocking notion converges to a stable matching. Knuth (1976) gives an example showing that a sequence of successive myopic blockings may form a cycle and never reach a stable matching.

Roth and Vande Vate (1990) show that, if a blocking pair is chosen randomly at each step of the process, the process reaches a stable matching with probability one. The sequential blocking process implicity assumes that no commitment is made by the agents. The critical feature of the process is that the agents are myopic: in each step, the acting blocking pair behaves as if the process terminates in their turn.

Our dynamic matching game also brings some insights from the literature of dynamic coalition-formation games [see, e.g., Chatterjee, Dutta, Ray, and Sengupta (1993), Bloch (1996), Ray and Vohra (1999), Bloch and Diamantoudi (2007)] to the study of decentralized matching market. With a dynamic game of coalition formation, in each period a proposer is selected among the set of active players using a protocol (often exogenously given) to make a proposal of forming a coalition and prospective members respond sequentially to such a proposal. This coalition is formed if all prospective members accept the proposal. Therefore, in each period, at most one coalition can form. In contrast, in our dynamic matching game, all active firms can simultaneously make offers and several firm-worker pairs can form in a single period. Another feature of our game is the different commitment structures that naturally arise in a labor market.

In the next section, we introduce a standard static model of matching. Section 3 defines our dynamic game of matching. Sections 4–6 study respectively the three aforementioned commitment structures. A short conclusion follows. The appendix contains some details omitted from the main text.

2 Static Matching Problem

In a static *matching problem*, introduced by Gale and Shapley (1962), there are two disjoint finite sets F and W of *firms* and *workers*. An *agent* refers to either a firm or a worker. Each agent i has a utility function u_i such that

$$u_f\colon W \cup \{f\} \to \mathbb{R} \quad \text{for all } f \in F,$$
$$u_w\colon F \cup \{w\} \to \mathbb{R} \quad \text{for all } w \in W.$$

Here, $u_i(j)$ denotes agent i's utility of being matched with agent j; $u_i(i)$ is the agent's utility of being unmatched. We normalize utilities so that $u_i(i) = 0$ for all i. If $u_i(j) \geq 0$, we say j is *acceptable* to i. We assume strict preferences: $u_i(j) = u_i(k)$ only if $j = k$.

For simplicity, we assume that each firm has only one position. A *matching* is then a function $\mu\colon F \cup W \to F \cup W$ such that (i) for all $f \in F$, $\mu(f) \in W \cup \{f\}$, (ii) for all $w \in W$, $\mu(w) \in F \cup \{w\}$, and (iii) for all $i, j \in F \cup W$, if $\mu(i) = j$ then $\mu(j) = i$. Here, $\mu(i)$ denotes the agent with whom i is matched. If $\mu(i) = i$, then i is not matched with anyone. Let $\mu_\emptyset$ denote the matching in which no one is matched.

A matching μ is *individually rational* if, for all $i \in F \cup W$, $\mu(i)$ is acceptable to

i. A matching μ is *blocked* by a pair $(f, w) \in F \times W$ if

$$u_f(w) > u_f(\mu(f)),$$
$$u_w(f) > u_w(\mu(w)).$$

That is, f and w both prefer each other to their partners under μ. A matching μ is *stable* if it is individually rational and has no blocking pair. By Gale and Shapley (1962), a stable matching exists for any matching problem.

3 Dynamic Matching Game

We consider a situation where agents are matched every period in a decentralized fashion. A *dynamic matching game*, parameterized by a list $(F, W, (u_i, \delta_i)_{i \in F \cup W}, F_c, W_c)$, is defined as follows.

3.1 Periods and Payoffs

Time periods are discrete and indexed by $t = 1, 2, 3, \ldots$. For academic labor markets, think of a period as one year. In each period, agents derive a payoff from the realized matching. The period-payoff function for agent i is u_i introduced above and is time-invariant. Each agent i maximizes the discounted sum of period-payoffs,

$$\sum_{t=1}^{\infty} \delta_i^{t-1} u_i(\mu^t(i)),$$

where μ^t is the realized matching in period t and $\delta_i \in (0, 1)$ is the discount factor.

3.2 Active Agents

At the beginning of each period $t = 1, 2, \ldots$, all agents observe the matching realized in the previous period, denoted by μ^{t-1}. We assume $\mu^0 = \mu_\emptyset$: no one is matched before the initial period.

The matching μ^{t-1} determines the set of firms and workers who are not able to move in period t. The set of *inactive firms* in period t is given by $F_c(\mu^{t-1}) \subseteq F$. These firms have committed themselves to their employees in μ^{t-1}. During period t, therefore, they can neither dismiss their employees nor hire new ones. That is, their current employees have tenure and there jobs are protected. Its complement, $F \setminus F_c(\mu^{t-1})$, is the set of *active firms*, which have not made any commitment to any worker. Therefore, active firms retain the right to dismiss their current employees if they have any. Similarly, let $W_c(\mu^{t-1})$ denote the set of *inactive workers* in period t, who cannot switch their employers in period t. The complement, $W \setminus W_c(\mu^{t-1})$, is the set of *active workers*, who have no commitment and can leave their current employers if they have been employed.

We consider the following three specifications for F_c and W_c.

Case 1: *No commitment*. All firms and workers are active regardless of the previous matching: $F_c(\mu^{t-1}) = W_c(\mu^{t-1}) = \emptyset$. Thus, firms can dismiss their employees, and workers can leave their current employers. That is, all labor contracts expire in one period.

Case 2: *Two-sided commitment*. All matched agents are inactive:

$$F_c(\mu^{t-1}) = \{f \in F : \mu^{t-1}(f) \neq f\},$$
$$W_c(\mu^{t-1}) = \{w \in W : \mu^{t-1}(w) \neq w\}.$$

Thus, once a firm and a worker are matched, they stay together permanently.

Case 3: *One-sided commitment*. All the matched firms are inactive, while all workers are active:

$$F_c(\mu^{t-1}) = \{f \in F : \mu^{t-1}(f) \neq f\},$$
$$W_c(\mu^{t-1}) = \emptyset.$$

Thus, workers cannot be dismissed but they may switch to other firms.

3.3 Period-Game

In every period, the agents play the following two-stage game.

In the first stage, every firm simultaneously makes an offer to at most one worker. An active firm can make an offer to any worker while an inactive firm has no option but to keep its employee under μ^{t-1}. For convenience, we treat inactive firms as if they make new offers to their current employees (i.e., renewal offers). Thus, firm f's action, denoted by o_f, is constrained by

$$\begin{cases} o_f \in W \cup \{f\} & \text{if } f \notin F_c(\mu^{t-1}), \\ o_f = \mu^{t-1}(f) & \text{if } f \in F_c(\mu^{t-1}), \end{cases}$$

where $o_f = f$ means that f makes no offer to any worker. Let $O_f(\mu^{t-1})$ denote the set of admissible actions for f.

In the second stage, each worker w privately observes the offers made to her in the first stage, denoted $O_w \equiv \{f \in F : o_f = w\}$. As noted above, O_w includes the renewal offer from the current employer if w has tenure. Workers do not observe any offer made to other workers in the current period. As noted above, each worker observes the entire matching realized in previous periods. Given these observations, each worker simultaneously accepts at most one offer. Active workers w can accept any offer or reject all. Inactive workers have no choice but to accept the renewal offer from their current employer. Thus, worker w's response, denoted by r_w, is constrained by

$$\begin{cases} r_w \in O_w \cup \{w\} & \text{if } w \notin W_c(\mu^{t-1}), \\ r_w = \mu^{t-1}(w) & \text{if } w \in W_c(\mu^{t-1}). \end{cases}$$

Let $R_w(\mu^{t-1}, O_w)$ denote the set of admissible responses for w.

Given the actions of firms and workers, the matching in period t, denoted μ^t, is determined by

$$\mu^t(w) = r(w) \quad \text{for all } w \in W.$$

3.4 Histories and Strategies

A history for firm f at the beginning of period t is an ordered list

$$h_f^t = (\mu^0 = \mu_\emptyset, o_f^1, \mu^1, o_f^2, \ldots, o_f^{t-1}, \mu^{t-1}),$$

where o_f^τ is the offer that f made in period τ and μ^τ is the matching realized in that period. While μ^τ is public information, o_f^τ is private information. Let H_f^t denote the set of histories for f at the beginning of period t. Let $H_f \equiv \cup_{t=1}^\infty H_f^t$ be the set of all histories for f.

A strategy of firm f is a function $\sigma_f \colon H_f \to \Delta(W \cup \{f\})$ such that for all $h_f^t \in H_f$, $\sigma_f(h_f^t) \in \Delta(O_f(\mu^{t-1}))$, where μ^{t-1} is the last entry of h_f^t.

Similarly, a history for worker w at period t (when she makes a decision) is an ordered list

$$h_w^t = (\mu^0 = \mu_\emptyset, O_w^1, r_w^1, \mu^1, O_w^2, \ldots, \mu^{t-1}, O_w^t),$$

where O_w^τ is the set of offers made to w in period τ (including a renewal offer if any) and r_w^τ is her reply in that period. Let H_w^t denote the set of all histories for w at period t. Let $H_w \equiv \cup_{t=1}^\infty H_w^t$ be the set of all histories for w. A strategy of worker w is then a function $\sigma_w \colon H_w \to \Delta(F \cup \{w\})$ such that, for all $h_w^t \in H_w$, $\sigma_w(h_w^t) \in \Delta(R_w(\mu^{t-1}, O_w^t))$, where μ^{t-1} and O_w^t are the last two entries of h_w^t.

A strategy profile $\sigma = (\sigma_i)_{i \in F \cup W}$ determines the expected payoff for each agent in the dynamic game. We limit ourselves to sequential equilibria in stationary strategies, where each agent's strategy depends only on the payoff-relevant state of the game, as we now define formally.

3.5 Stationary Strategies

In our dynamic matching game, the state variable is the matching in the previous period. However, distinct matchings may induce the same continuation game, depending on the commitment structure. We write $\mu \sim \mu'$ if μ and μ' induce the same continuation game, and we say that μ and μ' are *continuation equivalent*. The equivalence relation $\sim$ depends on the commitment structure of the game as follows.

In the no-commitment case, all matchings are continuation equivalent: $\mu \sim \mu'$ for all μ, μ'. In the absence of commitment, the continuation game is the same regardless of what happened in the previous periods.

In the two-sided commitment case, two matchings are continuation equivalent if and only if the set of unmatched agents is identical: $\mu \sim \mu'$ if and only if $\{i \in F \cup W : \mu(i) = i\} = \{i \in F \cup W : \mu'(i) = i\}$. The agents who have been matched cannot change their partner in the rest of the game. So what matters for the remaining agents is the set of remaining agents. How the matched agents are matched is irrelevant.[1]

In the one-sided commitment case, no two matchings are continuation equivalent: $\mu \sim \mu$ if and only if $\mu = \mu'$. Even if the set of matched agents is the same, the continuation game depends on how the agents are currently matched.

With the equivalence relation, we can define stationary strategies as follows. A firm f's strategy σ_f is *stationary* if for any two histories $h_f = (\ldots, \mu)$ and $h'_f = (\ldots, \mu')$ (possibly with different lengths), if $\mu \sim \mu'$ then $\sigma_f(h_f) = \sigma_f(h'_f)$. For workers' strategies, there is another requirement saying that the set of offers received in the current period is also identical. That is, a worker w's strategy σ_w is *stationary* if for any two histories $h_w = (\ldots, \mu, O_w)$ and $h'_w = (\ldots, \mu', O'_w)$, if $\mu \sim \mu'$ and $O_w = O'_w$ then $\sigma_w(h_w) = \sigma_w(h'_w)$. A *stationary equilibrium* is a sequential equilibrium in which everyone's strategy is stationary.

4 When No One Commits

We first consider the case where no one makes any commitment. The following result shows that, in the absence of commitment, the static notion of stability does a good job in predicting the outcome of stationary equilibrium.

Proposition 1 *Consider any dynamic matching game with no commitment. Then for any stationary equilibrium, the realized matching is identical in all periods and is stable. Conversely, for any stable matching, there exists a stationary equilibrium that yields this matching every period.*

In the absence of commitment, what happened in the previous periods is payoff-irrelevant and therefore is ignored by agents in stationary equilibria. Stated differently, what happens in the current period does not affect the outcome in the future. Because of this independence, agents can disregard the future and behave as in the static model.

A useful fact is that, for a fixed preference profile, the set of unmatched agents is identical in all stable matchings (Roth and Sotomayor, 1990). If an agent (worker or firm) is alone in one stable matching, he is alone in every stable matching. Therefore all the stationary equilibria make no difference for this agent.

5 When Both Sides Commit

We now consider the case where both sides of the market commit to their employment relationships. We first show that every stable matching is the outcome of some

[1] Technically speaking, two distinct matchings with the same set of unmatched agents induce different continuation games, since the matched agents' unique admissible action is labeled differently; where each matched firm has to make a renewal offer depends on the matching. But since the matched agents have no choice, we focus on the continuation game among the unmatched agents.

stationary equilibrium. Thus, no stable matching can be ruled out as an equilibrium outcome.

Proposition 2 *Consider any dynamic matching game with two-sided commitment. For any stable matching, there exists a stationary equilibrium that yields this matching every period.*

The proposition does not say anything about the outcomes in out-of-equilibrium subgames, but there actually exists a stationary equilibrium that induces a stable matching in every subgame. Specifically, for every subset of agents $S \subseteq F \cup W$, choose any stable matching μ^S for the subset. Given a collection $\{\mu^S\}_{S \subseteq F \cup W}$, we can show that there exists a stationary equilibrium such that, after any history, μ^S is formed where S is the set of active agents.

In Proposition 2, the equilibrium matching is determined completely in the initial period and there is no movement thereafter. The next result shows that stationary equilibria do not always have this feature.

Proposition 3 (Equilibria may involve a delay) *For some dynamic matching game with two-sided commitment, there exists a stationary equilibrium where some pairs are formed after the first period.*

Here is an example that shows why a delay is possible. There are 4 firms and 4 workers, whose ordinal preferences are given by

f_1	f_2	f_3	f_4	w_1	w_2	w_3	w_4
w_1	w_3	w_3	w_4	f_4	f_3	f_1	f_2
w_3	w_2	w_4	w_3	f_1	f_2	f_4	f_3
$\vdots$	w_4	w_1	w_2	f_3	f_4	f_2	f_1
	$\vdots$	w_2	w_1	f_2	f_1	f_3	f_4
		f_3	f_4	w_1	w_2	w_3	w_4

If the agents are sufficiently patient, there exists a stationary equilibrium in which $\{f_1, w_1\}$ and $\{f_2, w_2\}$ are matched in period 1 while $\{f_3, w_3\}$ and $\{f_4, w_4\}$ are matched in period 2. The complete description of the equilibrium is given in the appendix. Here we describe informally how the equilibrium works. In the initial period, every firm makes an offer. Firms f_1 and f_2 make offers to w_1 and w_2, respectively, and get accepted. Firms f_3 and f_4 also make offers to w_1 and w_2, respectively, and get rejected. In the second period, f_3 and f_4 make offers to w_3 and w_4, respectively, and get accepted.

The offers made by f_3 and f_4 in the first period are rejected in equilibrium. Why do not f_3 and f_4 make offers to w_3 and w_4 directly in the first period? The answer is simply that if f_3, for example, deviates in the first period and makes an offer to w_3, the offer will be rejected. As we will see shortly, the continuation equilibrium will match f_3 with w_4. Since f_3 prefers w_3, the deviation does not make it better off.

Why does w_3 reject f_3 when f_3 deviates? The key is that, as the recipient of f_3's offer, w_3 knows that w_1 did not receive an offer from f_3. Without an offer from f_3, the equilibrium prescribes w_1 to reject f_1. We will see why w_1 behaves in this way, but first let's see why w_3 rejects f_3. If w_3 rejects f_3, the behavior of w_1 implies that the only $\{f_2, w_2\}$ is matched in the current period. In the next period, the set of active agents is $F \cup W \setminus \{f_2, w_2\}$ and the continuation strategy prescribes they are matched immediately as $[\{f_1, w_1\}, \{f_3, w_4\}, \{f_4, w_3\}]$, which is a stable matching for the 6 agents. In the matching, w_3 gets her second choice, while f_3 is only her fourth choice. If w_3 is patient enough, therefore, she prefers to wait for her second choice.

The preceding paragraph does not explain why w_1 rejects f_1 in the event that w_1 did not receive an offer from f_3. The key to the answer is that, in this particular event, w_1 believes that f_3 made an offer to w_4. Since offers are private information, w_1 does not know where f_3 made an offer or whether f_3 made an offer at all. In the sequential equilibrium we construct, w_1 holds the particular belief we described. The belief is not unreasonable since, for f_3, making an offer to w_4 is the second best response. The action is actually the best response if f_3 is sufficiently impatient. If f_3 indeed made an offer to w_4, the equilibrium prescribes w_4 to accept the offer. Therefore w_1 believes that if she rejects f_1, the set of active agents in the next period will be $\{f_1, f_4, w_1, w_3\}$ and the continuation strategy will prescribe the agents to be matched immediately as $[\{f_1, w_3\}, \{f_4, w_1\}]$. The outcome is a stable matching for the 4 agents, where both workers get their first choice. Thus, by rejecting f_1 in the initial period, w_1 can get her first choice in the next period. If w_1 is patient enough, therefore, she prefers to wait for her first choice.

The equilibrium outcome, where $\{f_i, w_i\}$ is matched for all i, is not a stable matching, being blocked by f_2 and w_3. This brings us to the following result.

Proposition 4 (An equilibrium matching may be unstable) *For some dynamic matching game with two-sided commitment, there exists a stationary equilibrium whose final matching is not stable.*

The question is why the blocking pair does not form. Why does not f_2 make an offer to w_3 in the first period? They prefer each other, and once they are matched, they commit to each other. The answer is that, if f_2 makes an offer to w_3, the offer will be rejected. In the continuation equilibrium, f_2 will be matched with w_2. Thus the deviation only delays the matching with the same worker. The question is then: why does w_3 reject f_2? The answer is that if w_3 rejects f_2, the set of active agents in the next period is $F \cup W \setminus \{f_1, w_1\}$ and the continuation strategy prescribes them to be matched as $[\{f_2, w_2\}, \{f_3, w_4\}, \{f_4, w_3\}]$, which is a stable matching for the 6 agents. In the matching, w_3 gets her second choice, while her blocking partner, f_2, is only her third choice. If w_3 is patient enough, therefore, she rejects the blocking partner in order to be matched with an even more desirable firm in the next period.

The above description shows that the incentives that support delay and instability in the equilibrium have a similar structure. Delay and instability prevail in equilibrium because a firm's attempt to deviate from the equilibrium to either get the same worker earlier or get together with the blocking partner is thwarted by a rejection from the worker. The worker rejects the firm since doing so affects the continuation play in the worker's favor. This is the case even though offers are observed privately and the rejected offer is a deviation from an equilibrium. Rejecting a private out-of-equilibrium offer can change the continuation play since the deviation by the firm also affects the behavior of the worker who did not receive an offer from this firm.

The results of delay and instability are not driven by our assumption that offers are private information. The results survive if offers are publicly observable. It suffices to modify the strategy profile so that, once a firm deviates in the first period, the continuation equilibrium selects the worker-optimal stable matching. Then, for example, if f_2 deviates and makes an offer to w_3, then all workers reject all offers since they will get their first choice in the next period.

The results do rely on the standard assumption that the continuation equilibrium can vary with the state in any way. There is no a priori restriction on the relationship between the current state and equilibrium selection for the continuation game. Although the assumption is standard, you may wonder whether there might be a reasonable restriction on how the continuation equilibrium relates to the state. We suggest one simple restriction and studies its implications in Appendix A.

6 When Only Firms Commit

We now turn to the one-sided commitment case. As in academic job markets for seniors, workers are protected but do not commit themselves to their employers. In stationary equilibria, the payoff-relevant state is the matching in the previous period.

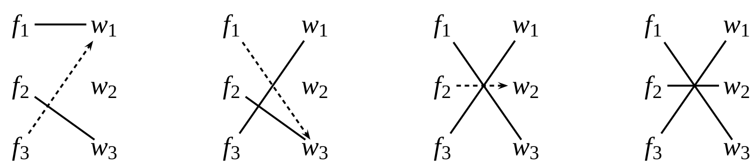
Proposition 5 *For some dynamic matching game with one-sided commitment, there exists a stationary equilibrium in which, in every period, the realized matching is unstable and such that every stable matching is a Pareto improvement for the workers.*

Proof. Here is an example that proves the result. There are 3 firms and 3 workers, whose ordinal preferences are given by

f_1	f_2	f_3	w_1	w_2	w_3
w_1	w_3	w_3	f_3	f_2	f_1
w_3	w_2	w_1	f_1	f_3	f_2
w_2	w_1	w_2	f_2	f_1	f_3
f_1	f_2	f_3	w_1	w_2	w_3

To simplify the exposition, we here assume that all workers are myopic: $\delta_w = 0$ for all w. The result itself holds even if workers are very patient, as we will discuss. By the assumption, workers simply take the best acceptable offer every period. We show that there exists a stationary equilibrium in which each f_i makes an offer to w_i in the initial period and the offers are accepted immediately. No firm can move thereafter. The outcome is not a stable matching since it is blocked by f_2 and w_3. The complete description of the equilibrium strategy profile is given in Figure 1. Here we highlight why the (unique) blocking pair does not get together.

To see why, suppose that f_2 makes an offer to w_3 in the first period. Since they are a blocking pair and workers are myopic, w_3 accepts. Therefore, in contrast to the two-sided commitment case, a firm's attempt to get together with its blocking partner does not receive an immediate rejection. However, since workers make no commitment, f_2 may lose w_3 later. The continuation equilibrium proceeds as depicted in the following figure.



The firm that can move in period 2 is f_3. Having failed to get its first choice, f_3 makes an offer to its second choice, w_1. The offer is accepted since f_3 is the first choice for w_1. In the next period, the only active firm is f_1. Having lost its first choice, f_1 makes an offer to its second choice, w_3. The offer is accepted since f_1 is the first choice for w_3. In period 4, f_2 has no choice but to make an offer to w_2 since the other workers are with their first choices. The matching is then completed. The result is actually the unique stable matching. The deviation by f_2 therefore helps the market reach a stable matching.

The question is whether f_2 gains from the deviation. While f_2 ends up with the same worker as in equilibrium, the deviation induces a different sequence of matchings. After the deviation, f_2 is matched with its first choice for two periods and has no worker for one period. The subsequent periods are not affected. Thus, f_2 does not gain from the deviation if and only if

$$(1 + \delta_{f_2})u_{f_2}(w_3) \leq (1 + \delta_{f_2} + \delta_{f_2}^2)u_{f_2}(w_2).$$

The inequality holds if $u_{f_2}(w_3) - u_{f_2}(w_2)$ is sufficiently small. Thus the deviation does not make the firm better off if the marginal gain from getting a better worker is sufficiently small. The same argument explains the incentives of f_2 in states $\{s10, s13, s21\}$ (see Figure 1). In this example, the deviation affects f_2 only in the short run: f_2 gets the same worker eventually. As we shall discuss shortly, we can also construct a similar example where the deviation makes f_2 worse off permanently: after the deviation, f_2 ends up with a worker who is less desirable than the worker it gets in equilibrium.

The incentives of f_2 in the other parts of the strategy are simple. The firm makes an offer to w_2 in $\{s4, s7, s15, s29\}$, but the reason is simply that w_3—the firm's first

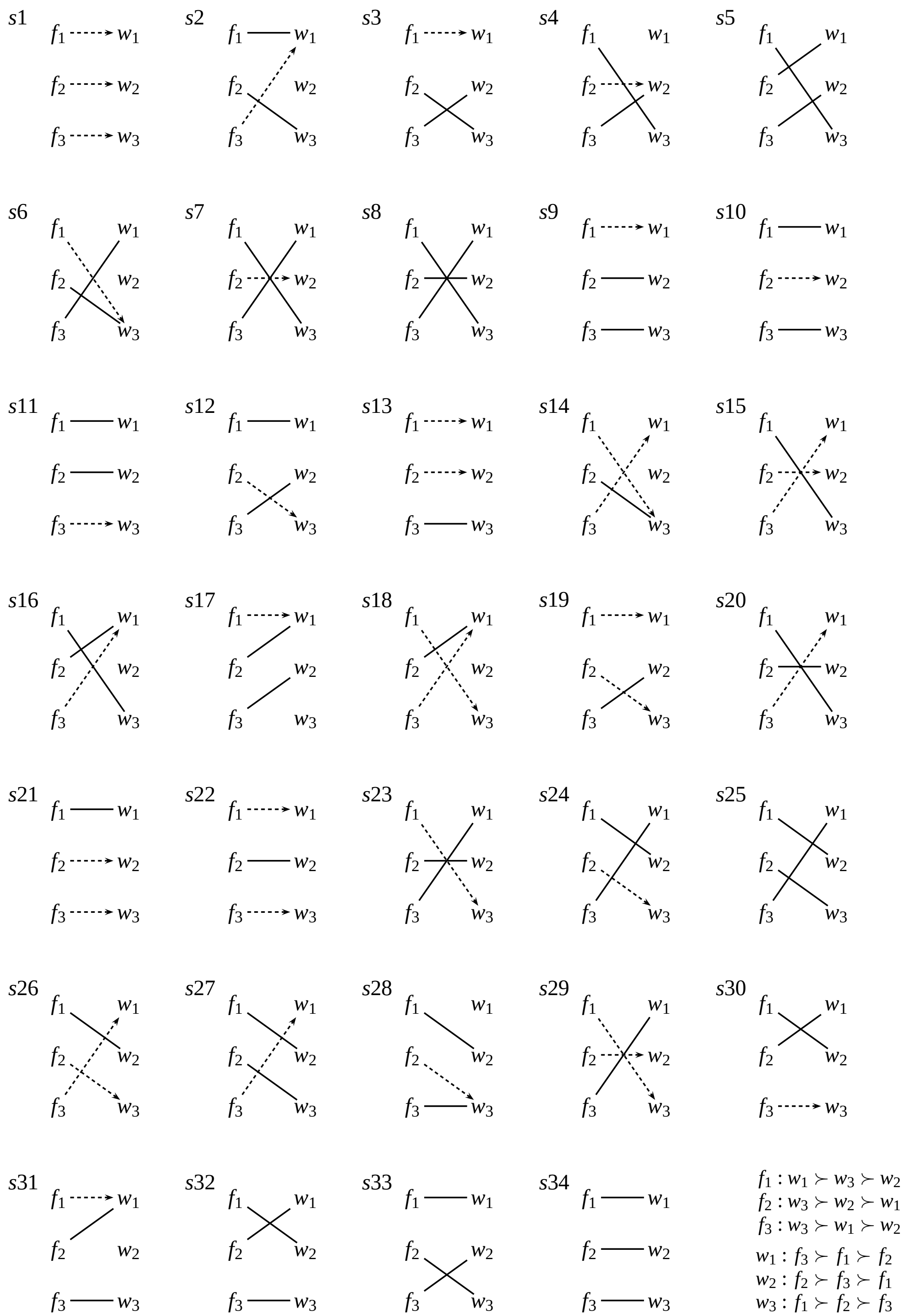


Figure 1: Firms' strategies in the proof of Proposition 5. The solid lines denote the matching at the beginning of the period and the dashed arrows denote the offers prescribed by our equilibrium strategies.

choice—has been or is being approached by her first choice. In the other states where f_2 can move, it makes an offer to the first choice and gets accepted.

The incentives of f_1 and f_3 are straightforward and no condition is necessary on their patience or payoff function. First, f_1 can always secure its second choice (w_3) since it is the first choice for the worker. In the equilibrium, f_1 does not make an offer to its first choice (w_1) only when the worker has been or is being approached by her first choice ($s6$, $s14$, $s18$, $s23$, and $s29$). Similarly, f_3 can always secure its second choice (w_1), but it is not liked by its first choice (w_3). In the equilibrium, f_3 does not make an offer to its first choice only when the first choice has been or is being approached by other firms ($s2$, $s14$–$s16$, $s18$, $s20$, $s26$, and $s27$). □

Intuitively, the unstable matching is sustained as equilibrium outcome since any attempt by f_2 to get the blocking partner succeeds only temporarily and backfires in the long run. The temporal success for f_2 in getting w_3 intensifies the competition among the firms. At the end, f_2 loses w_3 to f_1. If the loss from having no worker is large relative to the marginal gain from the better worker, the net effect of initiating a recruiting war is negative.

A comparison between the equilibrium and the unique stable matching ($s8$) reveals that none of the workers prefers the equilibrium outcome. While w_2 is indifferent, the other workers prefer the stable matching. This is interesting since workers are the ones who are protected by tenure. Without job protections, the stable matching is the unique equilibrium outcome. In this example, the workers are better off if job protections are removed.

The conclusion of Proposition 5 remains if workers and firms are very patient. In the Appendix, we construct an example showing that the proposition holds when $\delta_i \to 1$ for all $i \in F \cup W$. However, the construction is considerably more involved since we need to specify each worker's action contingent on every possible set of offers. Even with the small number of agents, constructing an equilibrium strategy profile is not easy since there are 34 states. The example is therefore relegated to Appendix B.4. In this example, if a firm deviates by making an offer to the blocking partner, it gets the worker only temporarily and ends up with a worker who is strictly less desirable than the permanent employee in the equilibrium.

If workers are myopic, we can also show that every stable matching can be supported as a permanent equilibrium outcome.[2] Whether the result extends to non-myopic workers is open at this point.

7 Conclusion

We considered a dynamic matching game in which firms and workers interact repeatedly in a decentralized job market. The main question was whether a decentral-

[2]Formally, consider any dynamic matching game with one-sided commitment where $\delta_w = 0$ for all w. Then for any stable matching, there exists a stationary equilibrium that yields this matching every period. See Appendix B.5 for the proof.

ized matching market generates a stable matching of Gale and Shapley (1962) and how the answer varies with the commitment structure of the market. Without commitment, we show that every stationary equilibrium matching is stable and every stable matching can be sustained by a stationary equilibrium. Once commitment is possible, this equivalence breaks down. It is possible for an equilibrium matching to be unstable. When only firms commit (i.e., there is job security), it is even possible that an equilibrium matching makes every worker worse off than any stable matching, illustrating the adverse effect job protection may have on workers' welfare.

To simplify our analysis, our dynamic matching game builds on the classical one-to-one matching model without monetary transfers (i.e., salary negotiations). Extending a many-to-one matching model with monetary transfers such as the one in Crawford and Knoer (1981) and Kelso and Crawford (1982) would certainly bring us closer to modeling real labor market dynamics. Recently, salary competition in matching model has been brought into focus by, for example, Bulow and Levin (2006) and Crawford (2006, forthcoming). It would be also interesting to make the commitment structure endogenous by letting players choose whether to commit.

Acknowledgments

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A Appendix: An Equilibrium Refinement for Two-Sided Commitment: Consistency

To motivate the refinement, suppose that, after a history, 3 firms and 3 workers are active, and the continuation equilibrium prescribes them to be matched as $[\{f_1, w_1\}, \{f_2, w_2\}, \{f_3, w_3\}]$. Suppose further that, for whatever reason, only pair $\{f_1, w_1\}$ was formed in the current period. Stationarity alone does not put any restriction on how the remaining four agents will be matched in the continuation game. However, one natural expectation is that the remaining agents will be matched as $[\{f_2, w_2\}, \{f_3, w_3\}]$ since it was the initial expectation. It appears to be a strong focal point. Why does anyone expect $[\{f_2, w_3\}, \{f_3, w_2\}]$?

The formalize the idea, let us limit ourselves to pure strategies. For any pure-strategy stationary strategy profile σ and any subset $S \subseteq F \cup W$, let $m(\sigma, S)$ denote the matching within S that realizes as the final result under σ in the continuation game where S is the initial set of active agents.

Definition A pure-strategy stationary equilibrium is *consistent* if for any subset $S \subseteq F \cup W$ and any subset $T \subseteq S$, if T is obtained from S by removing some matched pairs in $\mu \equiv m(\sigma, S)$ (i.e., for any firm $f \in S \setminus T$, $\mu(f)$ is a worker in $S \setminus T$), then $m(\sigma, T) = \mu|_T$.

Thus, if μ is the final matching in a continuation game, then after some of the pairs in μ are formed, the remaining agents are matched according to μ.

Proposition 6 *For any dynamic matching game with two-sided commitment and for any pure-strategy consistent stationary equilibrium, the final matching is stable.*

Proof. Let μ be the final matching (namely, $\mu = m(\sigma, F \cup W)$) and suppose that it is unstable. Let (f, w) be a blocking pair for μ. We choose a pair so that f is the most preferred firm to form a blocking pair with w. Then consider a subset $T = \{f, \mu(f), w, \mu(w)\} \cup \{i \in F \cup W : \mu(i) = i\}$. Within the subset, f is w's first choice among the firms for which w is acceptable. Therefore, if T is the set of active agents and f makes an offer to w, w will accept. So, in the continuation game, f gets w or someone better. However, consistency requires f to be matched with $\mu(f)$ in the continuation game. This is a contradiction since f prefers w to $\mu(f)$. $\square$

Proposition 7 *There exists a dynamic matching game with two-sided commitment where every pure-strategy stationary equilibrium is inconsistent.*

Proof. Consider a 5×5 matching problem with the following ordinal preferences:

f_1	f_2	f_3	f_4	f_5	w_1	w_2	w_3	w_4	w_5
w_1	w_2	w_3	w_4	w_5	f_2	f_1	f_1	f_4	f_3
w_3	w_1	w_5	$\vdots$	$\vdots$	f_3	f_2	f_3	f_3	f_5
w_2	$\vdots$	w_1			f_1	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$		w_4			$\vdots$				
		f_3							

The unique stable matching is where $\{f_i, w_i\}$ is formed for all i. Denote the matching by μ_1. Suppose, toward a contradiction, that there exists a pure-strategy consistent stationary equilibrium. Then, by Proposition 6, the final result of the equilibrium is μ_1. Consider a subset $S \equiv \{f_1, f_2, f_3, w_1, w_2, w_4\}$. Within the subset, there is a unique stable matching, which is $\mu_2 \equiv [\{f_1, w_2\}, \{f_2, w_1\}, \{f_3, w_4\}]$. Consider the continuation game where S is the initial set of active agents. Observe that the continuation equilibrium remains stationary and consistent. Therefore, by Proposition 6, the final result of the continuation equilibrium is μ_2 (i.e., $m(\sigma, S) = \mu_2$). Now, consider a subset $T \equiv \{f_1, f_2, w_1, w_2\}$. Since T is obtained from S by removing a matched pair in μ_2, consistency implies that in the continuation game where T is the initial set of active agents, the final result is $\mu_2|_T = [\{f_1, w_2\}, \{f_2, w_1\}]$. On the other hand, T is also obtained from the entire set of $F \cup W$ by removing three matched pairs in μ_1. Therefore, consistency also implies that the same continuation game for T results in $\mu_1|_T = [\{f_1, w_1\}, \{f_2, w_2\}]$. □

B Appendix: Proofs

B.1 Proof of Proposition 1

To prove the first statement, consider any stationary equilibrium. First, suppose, by way of contradiction, that the equilibrium yields an unstable matching μ^t in some period t. Let $\{f, w\}$ be a blocking pair for μ^t. Since the firms' strategies are stationary, w's action in period t does not affect the offers she will receive in the subsequent periods. This implies that, in period t, worker w's best action is to accept the most preferred acceptable offer. Since w prefers f to $\mu^t(w)$, it follows that w does not receive an offer from f in this period. Suppose then that, in period t, firm f deviates from the equilibrium and makes an offer to w. By the observation above, w will accept the offer and this has no influence over the other agents' strategies in the continuation game. Therefore, f gains from the deviation, a contradiction.

The above paragraph shows that the realized matching is stable in every period. Since firms' strategies are stationary, the realized matching is the same in every period.

To prove the converse, choose any stable matching μ and consider the following strategy profile: in every period, each firm f makes an offer to $\mu(f)$ and each worker w accepts the most preferred acceptable offer. The workers' strategies are optimal given that the firms' strategies are stationary. The firms' strategies are also optimal since making an offer to a preferred worker will be rejected since μ is stable, and making an offer to a less preferred worker will have no influence over the subsequent periods.

B.2 Proof of Proposition 2

For any subset $S \subseteq F \cup W$, let μ^S be any stable matching among S. Let σ be the strategy profile defined as follows. Consider any period and let S be the set of active agents. Each active firm f makes an offer to $\mu^S(f)$. Each active worker w who received an offer from $\mu^S(w)$ (and possibly others) accepts the most preferred offer. For each active worker w who did not receive an offer from $\mu^S(w)$, let T be defined by

$$T \equiv \{w, \mu^S(w)\} \cup O_w \cup \{i \in S : \mu^S(i) \in O_w \cup \{i\}\}. \tag{1}$$

Then w accepts the most preferred offer if

$$\max_{i \in O_w} u_w(i) > \delta_w u_w(\mu^T(w)). \tag{2}$$

Otherwise, w rejects all offers.

Let β be the belief system derived from the strategy profile above, with the following additional rule: in every period, if the set of active agents is S and an active worker w did not receive an offer from $\mu^S(w)$, then w believes that $\mu^S(w)$ did not make an offer to any worker.

We claim that (σ, β) is a sequential equilibrium. To see this, take any period and let S be the set of active agents.

We first examine firms' incentives. Let f be an active firm. If the firm follows the equilibrium strategy, it will be matched with $w \equiv \mu^S(f)$ in the current period (where $w = f$ is a possibility). If f makes an offer to any worker w' that f prefers to w, then since μ^S is a stable matching, the offer will be rejected. In the next period, therefore, the set of active agents will be $\{f, w\} \cup \{i \in S : \mu^S(i) = i\}$ and the best outcome for f is to get w. Since f gets w earlier in equilibrium, the firm does not gain from the deviation. Similarly, the firm does not gain by not making any offer.

Now, consider workers' incentives. Let w be an active worker. There are two cases. First, suppose that w receives an offer from $\mu^S(w)$, i.e., $\mu^S(w) \in O_w$. In this event, if w rejects all offers, the set of active firms in the next period will be

$$O_w \cup \{f \in F : \mu^S(f) = f\}.$$

Since μ^S is stable, w prefers $\mu^S(w)$ to any f such that $\mu^S(f) = f$ and for which w is acceptable. Therefore, if $\mu^S(w) \in O_w$, worker w gains nothing by waiting.

Suppose $\mu^S(w) \notin O_w$. By the definition of β, worker w believes that $\mu^S(w)$ did not make an offer to any worker. According to the belief, if w rejects all offers, the set of active agents in the next period will be T in (1) and hence the expected (average) utility is the right-hand side of (2). On the other hand, the maximum utility from accepting an offer in the current period is given by the left-hand side.

B.3 Proof of Propositions 3 and 4

Suppose that there are 4 firms and 4 workers and their ordinal preferences are given by

f_1	f_2	f_3	f_4	w_1	w_2	w_3	w_4
w_1	w_3	w_3	w_4	f_4	f_3	f_1	f_2
w_3	w_2	w_4	w_3	f_1	f_2	f_4	f_3
$\vdots$	w_4	w_1	w_2	f_3	f_4	f_2	f_1
	$\vdots$	w_2	w_1	f_2	f_1	f_3	f_4
		f_3	f_4	w_1	w_2	w_3	w_4

We now construct a stationary equilibrium (σ,β) under which $\{f_1, w_1\}$ and $\{f_2, w_2\}$ are matched in period 1 and $\{f_3, w_3\}$ and $\{f_4, w_4\}$ are matched in period 2. The final matching is blocked by f_2 and w_3.

In all periods where everyone is active (including the first period), firms make offers as follows:

$$f_1 \to w_1, \quad f_2 \to w_2, \quad f_3 \to w_1, \quad f_4 \to w_2.$$

Each worker w simply accepts the most preferred acceptable offer if the following condition holds:

$$\begin{aligned}
&O_{w_1} \cap \{f_1, f_4\} \neq \emptyset \text{ and } O_{w_1} \neq \{f_1\} && \text{if } w = w_1,\\
&O_{w_2} \cap \{f_2, f_3\} \neq \emptyset \text{ and } O_{w_2} \neq \{f_2\} && \text{if } w = w_2,\\
&O_{w_3} \neq \{f_2\}, \{f_3\} && \text{if } w = w_3,\\
&O_{w_4} \neq \{f_1\}, \{f_4\} && \text{if } w = w_4.
\end{aligned}$$

Otherwise, w rejects all firms.

For any proper subset $S \subsetneq F \cup W$, let μ^S be a stable matching within S. The choice of μ^S is arbitrary with the following exceptions:

S	μ^S
$\{f_3, f_4, w_3, w_4\}$	$[\{f_3, w_3\}, \{f_4, w_4\}]$
$\{f_1, f_4, w_1, w_3\}$	$[\{f_1, w_3\}, \{f_4, w_1\}]$
$\{f_2, f_3, w_2, w_4\}$	$[\{f_2, w_4\}, \{f_3, w_2\}]$
$F \cup W \setminus \{f_2, w_2\}$	$[\{f_1, w_1\}, \{f_3, w_4\}, \{f_4, w_3\}]$
$F \cup W \setminus \{f_1, w_1\}$	$[\{f_2, w_2\}, \{f_3, w_4\}, \{f_4, w_3\}]$

(3)

With a collection $\{\mu^S\}_{S\subsetneq F\cup W}$, we now specify the strategies in periods where the set of active agents is a proper subset $S \subsetneq F\cup W$, in the same way as in Proposition 2. So, firms make offers to their partners in μ^S. A worker w accepts the most preferred offer if the offer from the expected firm ($\mu^S(w)$) arrived. If the expected offer did not arrive to worker w, then let

$$T \equiv \{w, \mu^S(w)\} \cup O_w \cup \{i \in S : \mu^S(i) \in O_w \cup \{i\}\}.$$

Then w accepts the most preferred offer if $\max_{i\in O_w} u_w(i) > \delta_w u_w(\mu^T(w))$, and rejects all offers otherwise.

Let β be the belief system derived from the strategy profile defined above, with the following two additional rules. First, in periods where everyone is active and the set of offers made to worker w_1 is $O_{w_1} = \{f_1\}$ (thus the expected offer from f_3 did not arrive), then w_1 believes that f_3 made an offer to w_4. Similarly, in periods where everyone is active and the set of offers made to worker w_2 is $O_{w_2} = \{f_2\}$, then w_2 believes that f_4 made an offer to w_3. Second, in all other cases where a worker w was expecting an offer from a firm f but the offer did not come, the worker believes that f did not make an offer to any worker.

We now show that (σ, β) is a sequential equilibrium. Since the only difference from Proposition 2 is when everyone is active, we only check incentives in this state.

Firm f_1 has no incentive to deviate since it gets its first choice in equilibrium. Firm f_2 gets only its second choice (w_2) but does not gain by deviating. Indeed, if f_2 makes an offer to w_3 or w_1, this offer will be rejected and the firm will be matched with w_2 in the next period.[3] Similarly, by not making any offer, f_2 only delays its matching with w_2. Finally, if f_2 makes an offer to w_4, this offer will be accepted, which is not good for f_2 since it prefers w_2.

Firm f_3, on the other hand, gets his first choice (w_3) only in the next period. If this firm is patient enough (i.e., $\delta_{f_3} > u_{f_3}(w_4)/u_{f_3}(w_3)$), therefore, the only possible reason to deviate is to get the first choice in the current period. However, if f_3 makes an offer to w_3, then $O_{w_3} = \{f_3\}$ and hence the offer will be rejected. A symmetric argument applies to f_4.

For workers' incentives, we start with w_1 and w_2. Since they are symmetric, we need only to consider w_1. If she receives an offer from f_4, her optimal action is to accept the offer since f_4 is her top choice. So, in what follows, suppose that w_1 did not receive an offer from f_4. We divide the remaining case into two.

Suppose $O_{w_1} \neq \{f_1\}$. Then it can be checked that w_1 believes that if she rejects all offers, she will get the second choice (f_1) in the next period.[4] So, the optimal choice for w_1 depends on whether $f_1 \in O_{w_1}$. If $f_1 \in O_{w_1}$, then w_1 should accept f_1 in the current period since it is the best offer at hand and rejecting all offers will

[3] The set of active players in the next period will be $F \cup W \setminus \{f_1, w_1\}$.

[4] For example, suppose $O_{w_1} = \{f_1, f_2\}$. Then w_1 believes that f_3 did not make any offer and f_4 made an offer to w_2. According to w_2's strategy, w_2 will reject the offer from f_4. So if w_1 rejects all offers, she believes that, in the next period, everyone will be active and she will get f_1.

only delay the matching with the same firm. If $f_1 \notin O_{w_1}$, on the other hand, the optimal reply depends on the firm's patience. If the firm is sufficiently patient (i.e., $\delta_{f_1} > u_{w_1}(f_3)/u_{w_1}(f_1)$), the optimal reply is to reject all offers now and get f_1 in the next period.

Now, suppose $O_{w_1} = \{f_1\}$. By the construction of β, w_1 believes that f_3 made an offer to w_4 and this offer will be accepted. Thus, w_1 believes that if she rejects f_1, the set of active agents in the next period will be $\{f_1, f_4, w_1, w_3\}$ and hence she will get f_4, which is her first choice. If w_1 is patient enough, therefore, she prefers to wait for her first choice.

Finally, consider w_3 and w_4. Since they are symmetric, we only discuss w_3. If she receives an offer from her top choice (f_1), she obviously accepts it. So, suppose that she did not receive an offer from f_1. If $O_{w_3} = \{f_2\}$ and w_3 rejects the offer, the set of active agents in the next period will be $F \cup W \setminus \{f_1, w_1\}$ and w_3 will get her second choice (f_4). Since the offer at hand is her third choice, if w_3 is patient enough, she prefers to wait for her second choice. Similarly, if $O_{w_3} = \{f_3\}$ (which is the fourth choice for w_3) and w_3 rejects the offer, the set of active agents in the next period will be $F \cup W \setminus \{f_2, w_2\}$ and so w_3 will get her second choice (f_4).[5] So if w_3 is patient enough, she prefers to wait. If $f_4 \in O_{w_3}$, it can be checked that if w_3 rejects all offers, she will get either f_4 in the next period or f_3 in the following period. Since she prefers f_4 to f_3, she prefers to accept f_4 in the current period. Similarly, if $O_{w_3} = \{f_2, f_3\}$, rejecting all offers will give her f_3 in two periods, so she should accept f_3 in the current period.

B.4 Proof of Proposition 5 for Patient Workers

The proof of Proposition 5 in the main text relies on an example where $\delta_w = 0$ for all workers and therefore poses a question whether the result extends if the workers are patient. This section gives an example showing that the result does extend even if δ_w is close to 1 for all workers.

We consider a 3×3 matching problem with the following ordinal preferences:

f_1	f_2	f_3	w_1	w_2	w_3
w_1	w_3	w_3	f_2	f_3	f_1
w_3	w_2	w_1	f_3	f_2	f_2
w_2	w_1	w_2	f_1	f_1	f_3
f_1	f_2	f_3	w_1	w_2	w_3

[5] Note that w_1 will reject f_1.

Each agent's utility function is given by

$$u_i(j) = \begin{cases} 100 & \text{if } j \text{ is } i\text{'s first choice,} \\ 70 & \text{if } j \text{ is } i\text{'s second choice,} \\ 40 & \text{if } j \text{ is } i\text{'s third choice,} \\ 0 & \text{if } j \text{ is } i\text{'s last choice.} \end{cases}$$

There are two stable matchings:

$$[\{f_2, w_1\}, \{f_3, w_2\}, \{f_1, w_3\}],$$
$$[\{f_3, w_1\}, \{f_2, w_2\}, \{f_1, w_3\}].$$

In the equilibrium we construct, each firm f_i makes an offer to w_i, respectively, in the first period and they are all accepted. The realized matching, i.e., $[\{f_1, w_1\}, \{f_2, w_2\}, \{f_3, w_3\}]$, is not stable since it is blocked by $\{f_2, w_3\}$. Note that each of the stable matchings is a Pareto improvement for the workers.

Figure 2 describes the equilibrium strategy profile. For firms, the dashed arrows specify to whom each active firm makes an offer in the state. For workers, it is more complicated since a worker's response depends on not only the state but also the set of offers made to the worker. In the particular equilibrium we constructed, each worker's strategy in a given state s can be summarized by a *cutoff* denoted by $c_w(s, O) \in F \cup \{w\}$, where O is the set of offers made to the worker. The worker w simply chooses the most preferred offer that is at least as good as $c_w(s, O)$. In most cases, the cutoff is the worker herself, i.e., $c_w(s, O) = w$, which means that the worker chooses the most preferred acceptable firm. In the several cases where $c_w(s, O) \neq w$, the cutoffs are specified in Figure 2 in square brackets attached to the worker (e.g., $[f_2]$). Nothing is attached if the cutoff is oneself.

If the worker is currently employed, her current job is included in the offer set. Therefore, if the current job is less preferred to the cutoff, the worker resigns from the current job. For example, w_3 in $s25$ resigns from f_2 since f_2 is less preferred to f_1.

Except for worker 1 in states 6 and 14, the cutoffs are independent of the set of offers. In states 6 and 14, worker 1's cutoff depends on the number of offers (including the renewal offer).

To complete the description of the equilibrium, we need to specify workers' out-of-equilibrium beliefs since offers are private information. If a firm is prescribed to make an offer to a worker but deviates, this worker observes only the fact that the firm makes no offer to her. She does not observe where the firm makes an offer. In the particular equilibrium we construct, the worker in this situation is assumed to believe that the firm does not make any offer to any worker. The particular belief was chosen to simplify our construction of an equilibrium.

The strategy profile together with the belief system is a sequential equilibrium if the agents are sufficiently patient. A sufficient condition is that $\delta_i \geq \sqrt[3]{0.7} \approx 0.89$

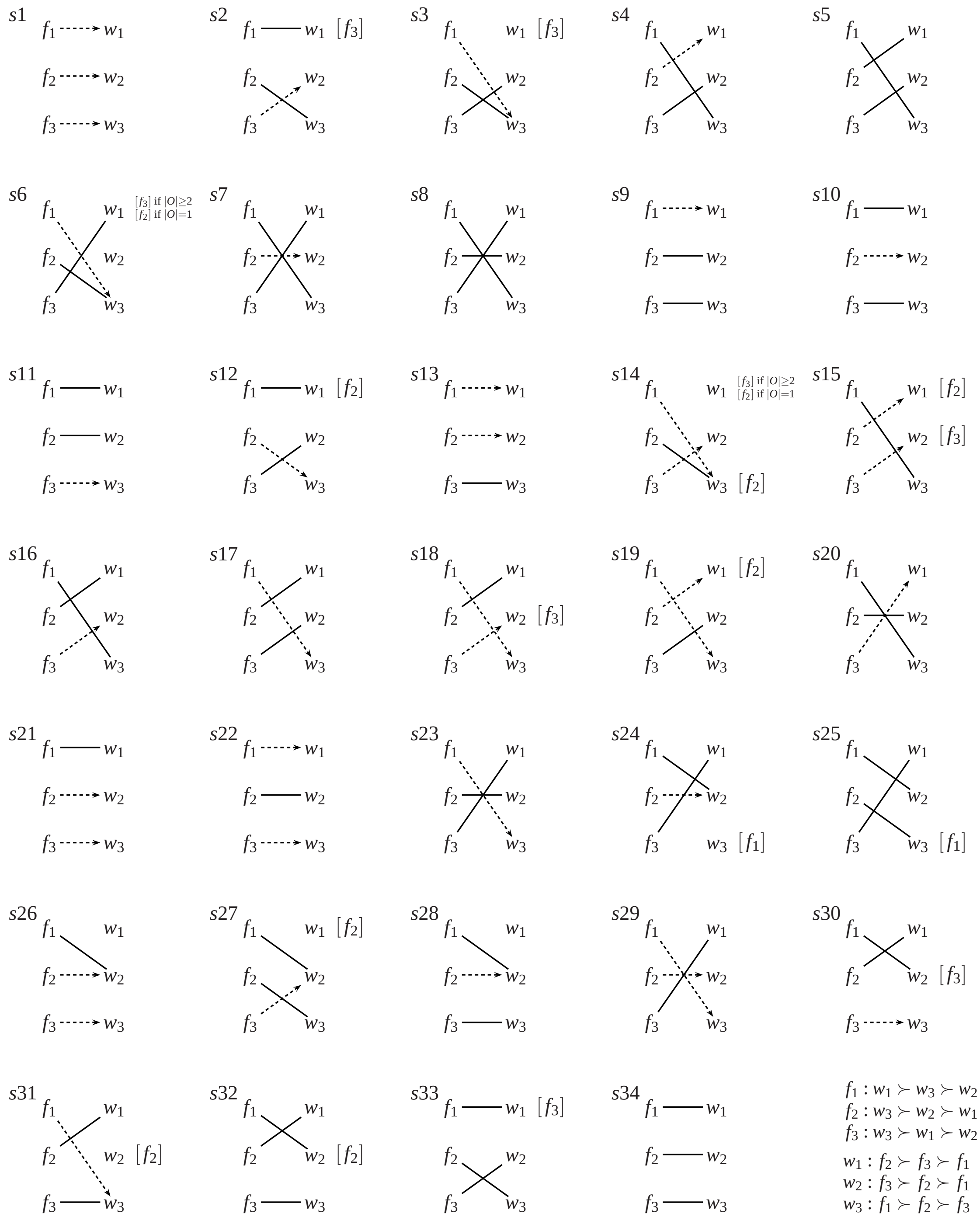
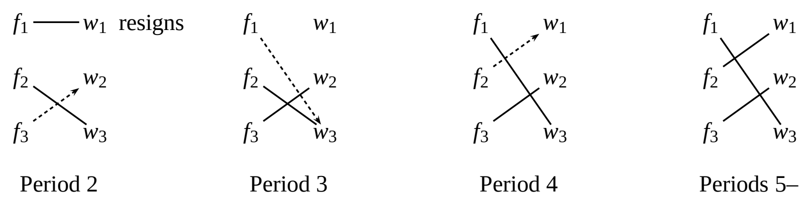


Figure 2: A stationary equilibrium that yields an unstable matching. The square brackets denote the cutoffs used by the workers.

for all agents. Verifying sequential rationality is extremely tedious. We therefore provide a computer program (a MATLAB code) that verifies sequential rationality when $\delta_i = 0.9$ for all i. Below we informally discuss why the blocking pair does not form.

If f_2 deviates in the first period and makes an offer to its blocking partner, w_3, then the offer would be accepted but trigger a chain of movements in subsequent periods as depicted in the following figure.



In the next period, the firm that moves is f_3. Having failed to get w_3, firm f_3 makes an offer to w_2 and gets accepted.[6] At the same time, w_1 resigns from f_1. This move by w_1 enables f_1 to make an offer to w_3 in period 3. The offer is accepted since f_1 is the first choice for w_3. In period 4, f_2 has no choice but to make an offer to w_1 since the other workers are with their first choice. The matching is then completed. Note that f_2, who initiates the process, ends up with a worker who is less desirable than the one the firm gets in equilibrium. Therefore, if f_2 is sufficiently patient, the deviation makes the firm worse off.

B.5 Proof of the Result in Footnote 2

As before, since workers are myopic and make no commitment, they simply accept the best acceptable offers every period. Since the state space and the action sets are finite, there exists a stationary equilibrium σ.[7] But the equilibrium outcome is unknown. Let μ be any stable matching. To support μ, we construct another strategy profile σ' as follows. If all agents are active, each firm f makes an offer to $\mu(f)$. If not all agents are active, follow σ. Thus, σ' differs from σ only when all agents are active. We shall show that σ' is a stationary equilibrium. By the construction of σ', it suffices to consider the state in which everyone is active. So, suppose that in period τ, everyone is active and a firm f deviates from σ. Since the outcome induced by the deviation may be stochastic, consider any path of play that occurs with a positive probability after the deviation. Let $\{w^t\}_{t=\tau}^{\infty}$ be the sequence of workers that f is matched with along the path (where $w^t \in W \cup \{f\}$). For any $t \geq \tau$, if $w^t \notin \{\mu(f), f\}$, then w^t prefers f to $\mu(w^t)$ since w^t receives an offer from $\mu(w^t)$ in period τ and a worker's period-payoff is non-decreasing over time. Since μ

[6] If f_3 deviates by making an offer to w_1, the offer will be accepted but the worker stays with the firm only for one period. After losing w_1, the firm will be eventually matched with w_2. The state transition is $s2 \rightarrow s6 \rightarrow s15 \rightarrow s5$. Given the specific utility function and discount rate, f_3 does not gain from the deviation.

[7] See, e.g., Mertens (2002).

is a stable matching, it follows that f prefers $\mu(f)$ to w^t. That is, along the path, f is never matched with a worker better than $\mu(f)$. Since this is the case for any possible path, f does not benefit from the deviation.

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