High Dimensional Consensus

A framework for distributed inference

José M. F. Moura

Work with: Usman A. Khan, Soummya Kar

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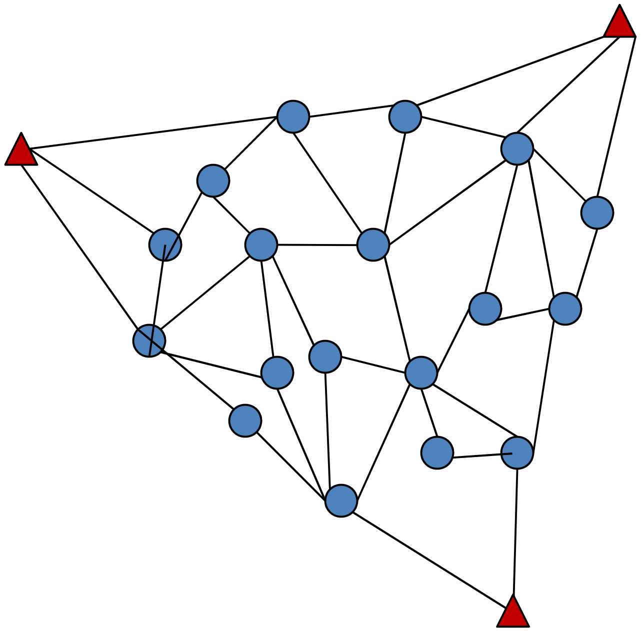
Distributed Algorithms: Sparse Networks



Distributed Algorithms

$$\mathbf{x}(t+1) = \left[1 - \alpha(t)\right] \mathbf{x}(t) + \alpha(t) \left\{ \mathbf{E} \odot \mathbf{P}\left(\overline{\mathbf{d}}_t\right) \left[\mathbf{x}(t) + \mathbf{v}(t)\right] + \mathbf{E} \odot \mathbf{B}\left(\overline{\mathbf{d}}_t\right) \left[\mathbf{u} + \mathbf{v}(t)\right] \right\}$$

- Graph randomness ($\mathbf{E}$)
- Graph randomness ($\mathbf{E}$)
- Sensors state ($\mathbf{x}(t+1)$)
- weights ($\alpha(t)$)
- weights ($\alpha(t)$)
- Graph sparsity ($\mathbf{P}$)
- Sensors state ($\mathbf{x}(t)$)
- Noise ($\mathbf{v}(t)$)
- Graph sparsity ($\mathbf{B}$)
- Anchors state ($\mathbf{u}$)
- Noise ($\mathbf{v}(t)$)

Distributed Algorithms

$$\mathbf{x}(t+1) = \left[1 - \alpha(t)\right]\mathbf{x}(t) + \alpha(t)\left\{\mathbf{E} \odot \mathbf{P}\left(\overline{\mathbf{d}}_t\right)\left[\mathbf{x}(t) + \mathbf{v}(t)\right] + \mathbf{E} \odot \mathbf{B}\left(\overline{\mathbf{d}}_t\right)\left[\mathbf{u} + \mathbf{v}(t)\right]\right\}$$

- **Average consensus:**

$$\mathbf{x}(t+1) = \mathbf{P}\mathbf{x}(t)$$

High Dimensional Consensus (HDC)

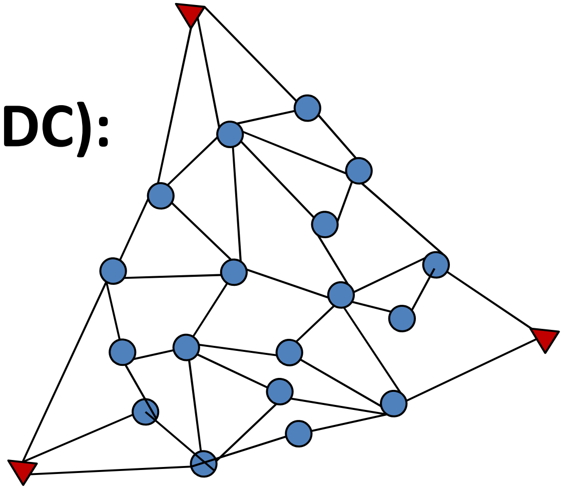
$$\mathbf{x}(t+1) = \left[1 - \alpha(t)\right]\mathbf{x}(t) + \alpha(t)\left\{\mathbf{E}\odot\mathbf{P}\left(\overline{\mathbf{d}}_t\right)\left[\mathbf{x}(t)+\mathbf{v}(t)\right] + \mathbf{E}\odot\mathbf{B}\left(\overline{\mathbf{d}}_t\right)\left[\mathbf{u}+\mathbf{v}(t)\right]\right\}$$

- **Average consensus:**

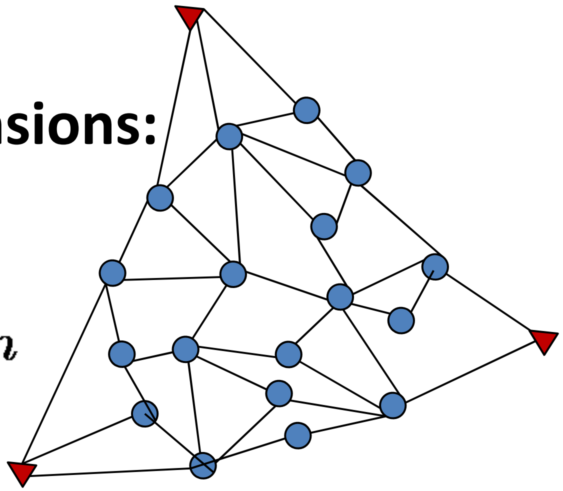
$$\mathbf{x}(t+1) = \mathbf{P}\mathbf{x}(t)$$

- **Deterministic High Dimensional Consensus (HDC):**

$$\begin{aligned}\mathbf{u}(t) &\equiv \mathbf{u}(0) = \mathbf{u}\\ \mathbf{x}(t+1) &= \mathbf{P}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)\end{aligned}$$



High Dimensional Consensus (HDC)



- *K* anchors and *M* sensors *(K+M=N)* in *m* dimensions:

$$\mathbf{U} = \left[\mathbf{u}^1(t)\cdots\mathbf{u}^m(t)\right], \mathbf{U}:\ K\times m$$
$$\mathbf{X}(t) = \left[\mathbf{x}^1(t)\cdots\mathbf{x}^m(t)\right], \mathbf{X}(t):\ M\times m$$

- Matrix HDC:

$$\mathbf{U}(t) = \mathbf{U}$$
$$\mathbf{X}(t+1) = \mathbf{P}\mathbf{X}(t) + \mathbf{B}\mathbf{U}$$

$$\begin{bmatrix}\mathbf{U}\\ \mathbf{X}(t+1)\end{bmatrix} = \begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{B} & \mathbf{P}\end{bmatrix}\begin{bmatrix}\mathbf{U}\\ \mathbf{X}(t)\end{bmatrix}$$

$$\lim_{t\to\infty}\mathbf{X}(t) = [\mathbf{I}-\mathbf{P}]^{-1}\mathbf{B}\mathbf{U}$$

Convergence

Lemma 1 *Let* $\mathbf{B} \neq 0$ *and* $\mathbf{U}(0) \notin \mathcal{N}(\mathbf{B})$, *where* $\mathcal{N}(\mathbf{B})$ *is the null space of* $\mathbf{B}$. *If*

$$\rho(\mathbf{P}) < 1,$$

then the limiting state of the sensors, $\mathbf{X}_\infty$, *is given by*

$$\mathbf{X}_\infty \triangleq \lim_{t\to\infty} \mathbf{X}(t+1) = (\mathbf{I} - \mathbf{P})^{-1}\mathbf{B}\mathbf{U}(0),$$

and the error, $\mathbf{E}(t) = \mathbf{X}(t) - \mathbf{X}_\infty$, *decays exponentially to* $\mathbf{0}$ *with exponent* $\ln(\rho(\mathbf{P}))$, *i.e.,*

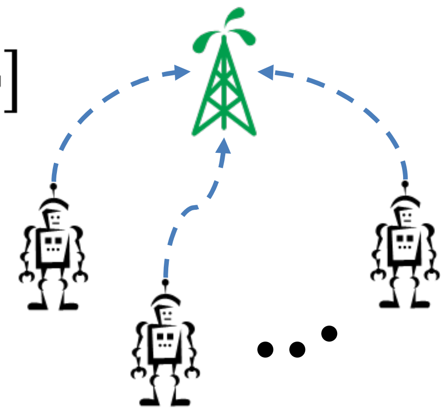
$$\limsup_{t\to\infty} \frac{1}{t}\ln\|\mathbf{E}(t)\| \ \leq \ \ln(\rho(\mathbf{P})).$$

Proof:

$$\mathbf{X}(t+1) \ = \ \mathbf{P}^{t+1}\mathbf{X}(0) + \sum_{k=0}^{t}\mathbf{P}^k\mathbf{B}\mathbf{U}(0),$$

Leader Follower

- *1* anchor in *m* dimensions: *K=1*, $\mathbf{u} = \left[u^1(t) \cdots u^m(t)\right]$



- *M* sensors: $$\mathbf{X}(t) = \begin{bmatrix} x_1^1(t) & \cdots & x_1^m(t) \\ & \cdots & \\ x_M^1(t) & \cdots & x_M^m(t) \end{bmatrix}$$

- HDC: $$\begin{bmatrix} \mathbf{u} \\ \mathbf{X}(t+1) \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{b} & \mathbf{P} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{X}(t) \end{bmatrix}$$

$$\lim_{t\to\infty} \mathbf{X}(t) = [\mathbf{I} - \mathbf{P}]^{-1} \mathbf{b}\mathbf{u}$$

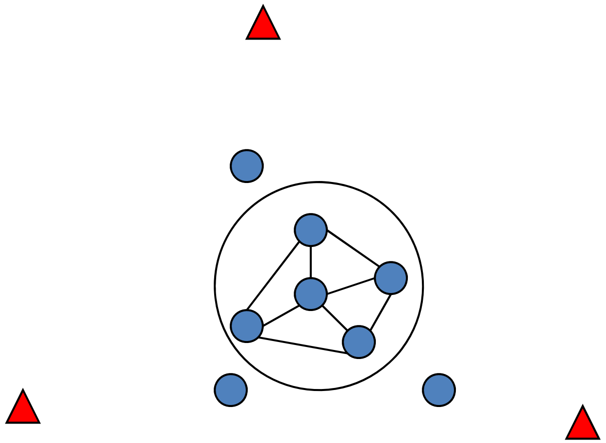
- In particular, for coordinate *j*: $$\lim_{t\to\infty} \mathbf{x}^j(t) = [\mathbf{I} - \mathbf{P}]^{-1} \mathbf{b} u^j$$

- Need: $$\begin{aligned} (\mathbf{I} - \mathbf{P})^{-1}\mathbf{b} &= \mathbf{1}_M, \\ \mathbf{b} + \mathbf{P}\mathbf{1}_M &= \mathbf{1}_M. \end{aligned} \quad \Rightarrow \quad \boxed{b_l + \sum_{i=1}^{M} p_{li}, l = 1, \cdots, M}$$

R. Olfati-Saber, J. A. Fax, and R. M. Murray. "Consensus and Cooperation in Networked Multi-Agent Systems," *Proceedings of the IEEE, vol. 95, no. 1, pp. 215-233, Jan. 2007.*

Distributed Localization

- **Localize *M* sensors with unknown locations in *m*-dimensional Euclidean space [1]**
- **Minimal number, $n=m+1$, of anchors with known locations**
- **Sensors only communicate in a neighborhood**
- **Only local distances in the neighborhood are known to the sensor**
- **There is no central fusion center**

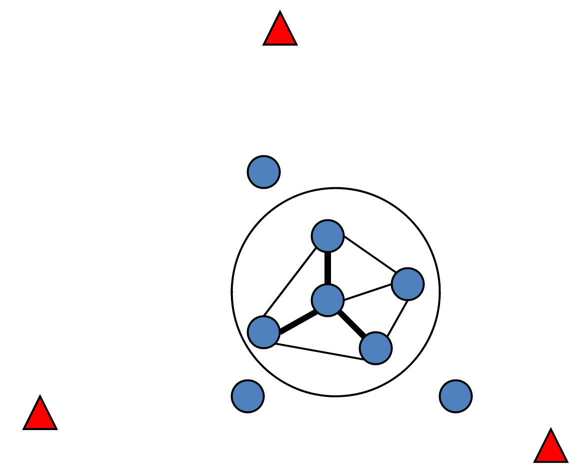


m = 2-D plane

[1] Khan, Kar, Moura, "Distributed Sensor Localization in Random Environments using Minimal Number of Anchor Nodes," *IEEE Tr. on Sign. Pr.*, 57(5), pp. 2000-2016, May 2009.

HDC: Distributed Sensor Localization

- Assumptions
  - Sensors lie in convex hull of anchors
  - Anchors not on a hyper-plane
  - Sensors find $m+1$ neighbors so they lie in their convex hull
  - Only local distances available

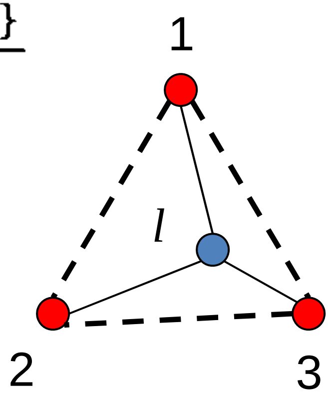


- Distributed localization (DILOC) algorithm
  - Sensor updates position estimate as convex l.c. of $n=m+1$ neighbors
  - Weights of l.c. are barycentric coordinates
  - Barycentric coordinates: ratio of generalized volumes
  - Barycentric coordinates: Cayley-Menger determinants (local distances)

$$\mathbf{x}_l(t+1) \quad = \quad p_{ll}\mathbf{x}_l(t) + \sum_{j \in \mathcal{K}_\Omega(l)} p_{lj}\mathbf{x}_j(t) + \sum_{k \in \mathcal{K}_\kappa(l)} b_{lk}\mathbf{u}_k(0).$$

- **Barycentric coordinates:** $p_{lk} = \frac{A_{\{l\}\cup\Theta_l\setminus\{k\}}}{A_{\Theta_l}}$
- **Example 2D:**

$$p_{l3} = \frac{A_{1l2}}{A_{123}}$$



- **Cayley-Menger determinants:**

$$A_\kappa^2 = \frac{1}{s_{m+1}} \begin{vmatrix} 0 & \mathbf{1}_{m+1}^T \\ \mathbf{1}_{m+1} & \mathbf{\Gamma} \end{vmatrix},$$

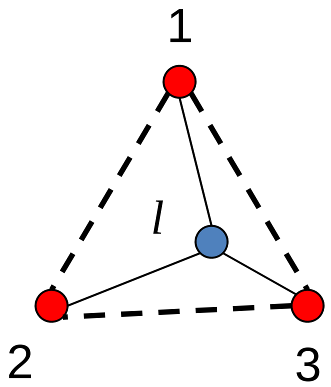
where $\mathbf{\Gamma} = \{d_{lj}^2\}$, $l, j \in \kappa$, is the matrix of squared distances, d_{lj}, among the $m+1$ points in κ and

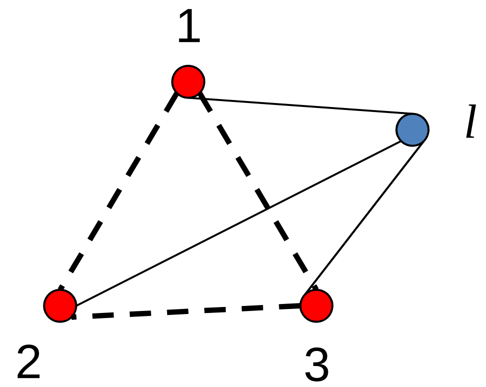
$$s_m = \frac{2^m (m!)^2}{(-1)^{m+1}}, \qquad m = \{0, 1, 2, \ldots\}.$$

Its first few terms are $-1, 2, -16, 288, -9216, 460800, \ldots$.

Set-up phase: Triangulation

- Test to find a triangulation set,
- Convex hull inclusion test: based on the following observation.





$$A_{l12} + A_{l13} + A_{l23} > A_{123} \qquad A_{l12} + A_{l13} + A_{l23} > A_{123}$$

- The test becomes

$$l \in \mathcal{C}(\Theta_l), \qquad \text{if} \sum_{k \in \Theta_l} A_{\Theta_l \cup \{l\} \setminus \{k\}} = A_{\Theta_l},$$

$$l \notin \mathcal{C}(\Theta_l), \qquad \text{if} \sum_{k \in \Theta_l} A_{\kappa \cup \{l\} \setminus \{k\}} > A_{\Theta_l}$$

Distributed Localization

- **Distributed localization algorithm (DILOC)**

$$\mathbf{x}_l(t+1) = p_{ll}\mathbf{x}_l(t) + \sum_{j \in \mathcal{K}_\Omega(l)} p_{lj}\mathbf{x}_j(t) + \sum_{k \in \mathcal{K}_\kappa(l)} b_{lk}\mathbf{u}_k(0).$$

- **Convergence:** $\sum_{k \in \Theta_l} A_{\Theta_l \cup \{l\} \setminus \{k\}} = A_{\Theta_l}, \quad l \in \mathcal{C}(\Theta_l)$

$$\Sigma_{k \in \Omega_l} p_{lk} + \Sigma_{k \in \kappa_l} b_l k = 1$$

Lemma 3: The underlying Markov chain with the transition probability matrix given by the iteration matrix $\mathbf{\Upsilon}$ is absorbing.

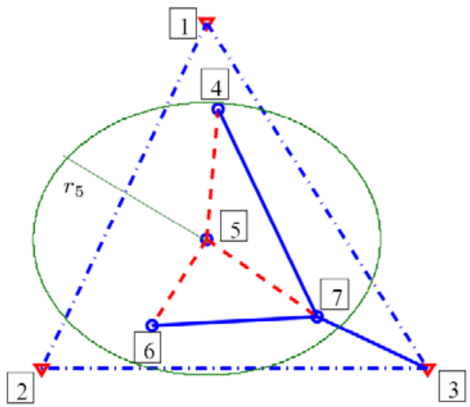
Theorem 1 (DILOC Convergence): DILOC (10) converges to the *exact* sensor coordinates, $\mathbf{c}_l^*, l \in \Omega$, i.e.,

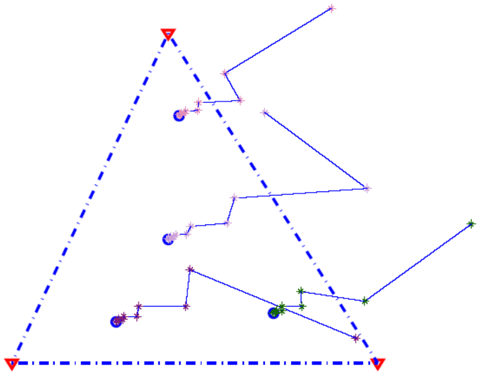
$$\lim_{t \to \infty} \mathbf{c}_l(t+1) = \mathbf{c}_l^* \quad \forall l \in \Omega. \tag{36}$$

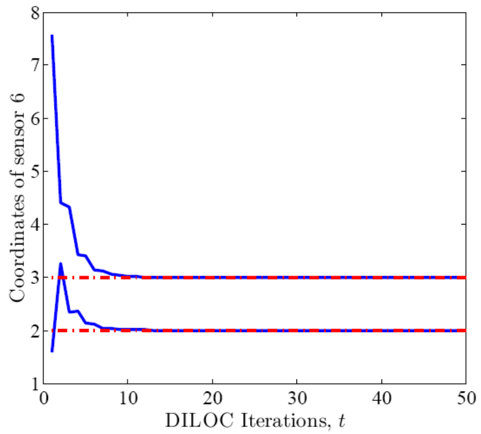
Distributed Localization: Simulations

- N=7 **node network in** 2**-d plane**

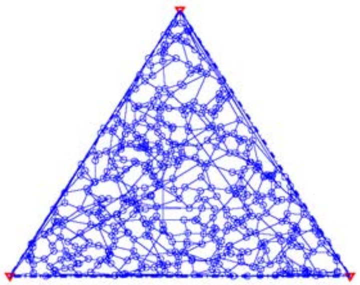
- M= 4 **sensors,** $K = m+1 = 3$ **anchors**

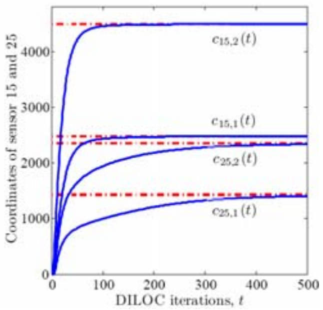






- M = 497 **sensors**





Random Network, Noisy Comm., Errors

- **Link failures (packet drops):**
  - **Links are modeled as Bernoulli random variables (temporally independent, possibly spatially correlated)** $e_{ln}(t), l \in \Omega, n \in \Theta_l$
  - $e_{ln}(t)$ **=** 0 (link failure) with probability $1 - q_{nl}$
- **Communication is noisy:**

$$y^j_{ln}(t) = x^j_n(t) + v^j_{ln}(t), \qquad n \in \Theta_l,$$

  - **Noise is zero mean, finite 2nd moment, no distributional assumptions**
- **Barycentric coordinates are noisy:**
  - **Intersensor distances are noisy induce perturbation of the barycentric coordinates**
- $\mathbf{P}\left(\widehat{\mathbf{d}}_t\right) = \mathbf{P}(\mathbf{d}^*) + \mathbf{S_P} + \widetilde{\mathbf{S}}_\mathbf{P}(t) \triangleq \{\widehat{p}_{ln}(t)\}$, $\mathbf{B}\left(\widehat{\mathbf{d}}_t\right) = \mathbf{B}(\mathbf{d}^*) + \mathbf{S_B} + \widetilde{\mathbf{S}}_\mathbf{B}(t) \triangleq \{\widehat{b}_{ln}(t)\}$

Distributed Localization

- **Theorem: Link failures, noisy comm., errors in intersensor distances – Under noise model, persistence cond., and connected on average**

$$\alpha(t) \geq 0, \sum_t \alpha(t) = \infty, \sum_t \alpha^2(t) < \infty.$$

$$L(i) = \overline{L} + \widetilde{L}(i),\ \forall i \geq 0\ , \quad \lambda_2\left(\overline{L}\right) > 0.$$

HDC for distance localization

$$\mathbf{x}_l(t+1) = (1-\alpha(t))\,\mathbf{x}_l(t) + \alpha(t)\left[\sum_{n\in\Omega\cap\Theta_l} \frac{e_{ln}(t)\widehat{p}_{ln}(t)}{q_{ln}}(\mathbf{x}_n(t)+\mathbf{v}_{ln}(t)) + \sum_{k\in\kappa\cap\Theta_l} \frac{e_{lk}(t)\widehat{b}_{lk}(t)}{q_{lk}}(\mathbf{u}_k+\mathbf{v}_{lk}(t))\right]$$

converges

$$\lim_{t\to\infty} \mathbf{X}(t+1) = (\mathbf{I}-\mathbf{P}-\mathbf{S_P})^{-1}(\mathbf{B}+\mathbf{S_B})\mathbf{U}(0)$$

Random Network

- **Link failures (packet drops):**
  - **Links are modeled as Bernoulli random variables (temporally independent, possibly spatially correlated)** $e_{ln}(t), l \in \Omega, n \in \Theta_l$
  - $e_{ln}(t)$ = 0 (link failure) with probability $1 - q_{nl}$
- **Communication is noisy:**

$$y_{ln}^{j}(t) = x_{n}^{j}(t) + v_{ln}^{j}(t), \qquad n \in \Theta_l,$$

  - **Noise is zero mean, finite 2nd moment, no distributional assumptions**
- **Intersensor distances are noisy:**

(B.3) **Noisy distance measurements:** Let $\{Z(t)\}_{t\geq 0}$ be any sequence of inter-node distance measurements collected over time. Then, there exists a sequence of estimates $\{\overline{\mathrm{d}}_t\}_{t\geq 0}$ such that, for all t, $\overline{\mathrm{d}}_t$ can be computed *efficiently* from $\{X(s)\}_{s\leq t}$ and we have

$$\mathbb{P}\left[\lim_{t\to\infty} \overline{\mathrm{d}}_t = \mathrm{d}^*\right] = 1 \tag{15}$$

$$\overline{d}_{ab}(t) = \frac{1}{t}\sum_{s\leq t} \widetilde{d}_{ab}(s) = \frac{t-1}{t}\overline{d}_{ab}(t-1) + \frac{1}{t}\widetilde{d}_{ab}(t), \qquad \overline{d}_{ab}(0) = \widetilde{d}_{ab}(0)$$

Distributed Localization

- **Theorem: Errors in intersensor dist. – Under noise model**

$$\mathbb{P}\left[\lim_{t\to\infty} \overline{\mathrm{d}}_t = \mathrm{d}^*\right] = 1$$

Persistence cond.

$$\alpha(t) \geq 0,\ \lim_{t\to\infty} \alpha(t) = 0,\ \text{and} \sum_t \alpha(t) = \infty,\quad \alpha(t) = \frac{a}{(t+1)^{\delta}},\ 0 < \delta \leq 1$$

HDC for distance localization: $\mathbf{X}(t) = \left[\mathbf{x}^1(t)\cdots\mathbf{x}^m(t)\right]$

$$\mathbf{x}^j(t+1) = (1-\alpha(t))\mathbf{x}^j(t) + \alpha(t)\left[\mathbf{P}\left(\overline{\mathrm{d}}_t\right)\mathbf{x}^j(t) + \mathbf{B}\left(\overline{\mathrm{d}}_t\right)\mathbf{u}^j\right],\ 1 \leq j \leq m$$

converges

$$\mathbb{P}\left[\lim_{t\to\infty} \mathbf{x}^j(t) = \left(\mathbf{I} - \mathbf{P}(\mathrm{d}^*)\right)^{-1}\mathbf{B}(\mathrm{d}^*)\mathbf{u}^j,\ \forall j = 1,\ldots,m\right] = 1$$

Proof

$$\mathbf{x}^j(t+1) \quad = \quad (1-\alpha(t))\mathbf{x}^j(t) + \alpha(t)\left[\mathbf{P}\left(\overline{\mathbf{d}}_t\right)\mathbf{x}^j(t) + \mathbf{B}\left(\overline{\mathbf{d}}_t\right)\mathbf{u}^j\right], \;\; 1 \leq j \leq m$$

Lemma 2 Consider the sequence of iterations in (16). We have

$$\mathbb{P}\left[\sup_{t\geq 0}\left\|\mathbf{x}^j(t)\right\| < \infty, \quad 1 \leq j \leq m\right] = 1.$$

In other words, the sequence $\{\mathbf{x}^j(t)\}_{t\geq 0}$ remains bounded a.s. for all j.

Proof of Lemma 2:

$$\mathbf{x}^j(t+1) \quad = [(1-\alpha(t)I) + \alpha(t)\mathbf{P}(\mathbf{d}^*)]\,\mathbf{x}^j(t) + \alpha(t)\left[\mathbf{P}(\overline{\mathbf{d}}_t) - \mathbf{P}(\mathbf{d}^*)\right]\mathbf{x}^j(t) + \alpha(t)\mathbf{B}(\overline{\mathbf{d}}_t)\mathbf{u}$$

$$\|(1-\alpha(t)\mathbf{I}) + \alpha(t)\mathbf{P}(\mathbf{d}^*)\|_P \leq 1 - \alpha(t) + \alpha(t)\,\|\mathbf{P}(\mathbf{d}^*)\|_P = 1 - \lambda^*\alpha(t)$$

$$\left\|\mathbf{P}(\overline{\mathbf{d}}_t) - \mathbf{P}(\mathbf{d}^*)\right\|_P \leq \varepsilon \;\;, \qquad \left\|B(\overline{\mathbf{d}}_t)\right\|_P \leq \lambda_1(\omega).$$

$$\begin{aligned}\left\|\mathbf{x}^j(t+1)\right\|_P &\leq (1-\lambda^*)\left\|\mathbf{x}^j(t)\right\|_P + \varepsilon\alpha(t)\left\|\mathbf{x}^j(t)\right\|_P + \alpha(t)\lambda_1(\omega)\left\|\mathbf{u}^j\right\|_P \\ &= (1-(\lambda^*-\varepsilon)\alpha(t))\left\|\mathbf{x}^j(t)\right\|_{\_} + \alpha(t)\lambda_1(\omega)\left\|\mathbf{u}^j\right\|_{\_}\end{aligned}$$

$$\begin{aligned}\left\|\mathbf{x}^j(t)\right\|_P &\leq \left(\prod_{k=t_2(\omega)}^{t-1}(1-a_1\alpha(t))\right)\|\mathbf{x}(t_2(\omega))\|_P + \sum_{k=t_2(\omega)}^{t-1}\left[\left(\prod_{l=k+1}^{t-1}(1-a_1\alpha(l))\right)a_2\alpha(k)\right] \\ &\leq \|\mathbf{x}(t_2(\omega))\|_P + \sum_{k=t_2(\omega}^{t-1}\left[\left(\prod_{l=k+1}^{t-1}(1-a_1\alpha(l))\right)a_2(\omega)\alpha(k)\right]\end{aligned}$$

Proof (continued)

$$\left\|\mathbf{x}^{j}(t)\right\|_{P} \leq \left(\prod_{k=t_2(\omega)}^{t-1}(1-a_1\alpha(t))\right)\|\mathbf{x}(t_2(\omega))\|_{P} + \sum_{k=t_2(\omega)}^{t-1}\left[\left(\prod_{l=k+1}^{t-1}(1-a_1\alpha(l))\right)a_2\alpha(k)\right]$$

$$\leq \|\mathbf{x}(t_2(\omega))\|_{P} + \sum_{k=t_2(\omega}^{t-1}\left[\left(\prod_{l=k+1}^{t-1}(1-a_1\alpha(l))\right)a_2(\omega)\alpha(k)\right]$$

Lemma 1 (Lemma 18, [9]) Let the sequences $\{r_1(t)\}_{t\geq 0}$ and $\{r_2(t)\}_{t\geq 0}$ be given by

$$r_1(t) = \frac{a_1}{(t+1)^{\delta_1}}, \qquad r_2(t) = \frac{a_2}{(t+1)^{\delta_2}} \tag{24}$$

where $a_1, a_2, \delta_2 \geq 0$ and $0 \leq \delta_1 \leq 1$. Then, if $\delta_1 = \delta_2$, there exists $K > 0$, such that, for non-negative integers, $s < t$,

$$0 \leq \sum_{k=s}^{t-1}\left[\prod_{l=k+1}^{t-1}(1-r_1(l))\right]r_2(k) \leq K \tag{25}$$

Moreover, the constant K can be chosen independently of s, t. Also, if $\delta_1 < \delta_2$, then, for arbitrary fixed s,

$$\lim_{t\to\infty}\sum_{k=s}^{t-1}\left[\prod_{l=k+1}^{t-1}(1-r_1(l))\right]r_2(k) = 0 \tag{26}$$

$\Longrightarrow$ $\sup_{t\geq 0}\left\|\mathbf{x}^{j}(t)\right\|_{P} \leq K(\omega)$ $\Longrightarrow$ $\mathbb{P}\left[\sup_{t\geq 0}\left\|\mathbf{x}^{j}(t)\right\|_{P} < \infty\right] = 1$

[9] S. Kar, J. M. F. Moura, and K. Ramanan, "Distributed parameter estimation in sensor networks: Nonlinear observation models and imperfect communication," Submitted for publication, see also http://arxiv.org/abs/0809.0009, Aug. 2008.

Proof of Theorem

Proof Theorem: Cannot use standard stochastic approx. techniques because

$$\mathbf{P}(\overline{\mathbf{d}}_t), \mathbf{B}(\overline{\mathbf{d}}_t)$$

are a function of past measurements, so strongly time dependent, non Markovian

We use a comparison argument. To this end, consider the idealized update

$$\widetilde{\mathbf{x}}^j(t+1) = (1-\alpha(t))\,\widetilde{\mathbf{x}}^j(t) + \alpha(t)\left[\mathbf{P}(\mathbf{d}^*)\widetilde{\mathbf{x}}(t) + \mathbf{B}(\mathbf{d}^*)\mathbf{u}^j\right]$$

$$\lim_{t\rightarrow\infty} \widetilde{\mathbf{x}}^j(t) = \left(\mathbf{I} - \mathbf{P}(\mathbf{d}^*)\right)^{-1}\mathbf{B}(\mathbf{d}^*)\mathbf{u}^j.$$

$$\{\mathbf{e}^j(t) = \mathbf{x}^j(t) - \widetilde{\mathbf{x}}^j(t)\}_{t\geq 0}$$

$$\mathbf{e}^j(t+1) = (1-\alpha(t))\,\mathbf{e}^j(t) + \alpha(t)\mathbf{P}(\mathbf{d}^*)\mathbf{e}^j(t) + \alpha(t)\left(\mathbf{P}(\overline{\mathbf{d}}_t) - \mathbf{P}(\mathbf{d}^*)\right)\mathbf{x}^j(t) + \alpha(t)\left(\mathbf{B}(\overline{\mathbf{d}}_t) - \mathbf{P}(\mathbf{d}^*)\right)\mathbf{u}^j.$$

$$\lim_{t\rightarrow\infty}\left\|\mathbf{e}^j(t)\right\|_P = 0$$

Localization: General Random, Noise, Errors

- **Theorem: Random network, noisy comm., errors in intersensor dist., connected on average**

$$\mathbf{x}^j(t+1) = (1-\alpha(t))\mathbf{x}^j(t) + \alpha(t)\left[\mathbf{E}\odot\mathbf{P}\left(\overline{\mathbf{d}}_t\right)\left(\mathbf{x}^j(t)+\mathbf{v}^j(t)\right) + \mathbf{E}\odot\mathbf{B}\left(\overline{\mathbf{d}}_t\right)\left(\mathbf{u}^j+\mathbf{v}^j(t)\right)\right]$$

Persistence cond. $\alpha(t)\geq 0, \sum_t \alpha(t)=\infty, \sum_t \alpha^2(t)<\infty.$

Connected $L(i)=\overline{L}+\widetilde{L}(i),\ \forall i\geq 0$ **,** $\lambda_2\left(\overline{L}\right)>0.$

HDC for distance localization converges

Proof: comparison argument

$$\widetilde{\mathbf{x}}^j(t+1) = (1-\alpha(t))\widetilde{\mathbf{x}}^j(t) + \alpha(t)\left[\mathbf{P}\left(\overline{\mathbf{d}}_t\right)\widetilde{\mathbf{x}}^j(t) + \mathbf{B}\left(\overline{\mathbf{d}}_t\right)\widetilde{\mathbf{u}}^j\right]$$

$\widetilde{\mathbf{e}}^j(t) = \mathbf{x}^j(t) - \widetilde{\mathbf{x}}^j(t)$ $\Longrightarrow$ $\left\{\widetilde{\mathbf{e}}^j(t)\right\}$ $\Longrightarrow$ **0, a.s.**

Consensus: Random Network, Noisy Comm.

$$\mathbf{x}(i+1) = \mathbf{x}(i) - \alpha(i)\left[L(i)\mathbf{x}(i) + \mathbf{n}(i)\right]$$

- **Laplacian indep. in time, noise iid: standard stoch. approx.**

  Limiting random variable θ:

$$\mathbb{E}[\theta] = r$$

$$\zeta \leq \frac{\eta}{N^2}\sum_{i\geq 0}\alpha^2(i)$$

  ***M* realizable links, identical prob. failures, noise iid var. σ^2**

$$\zeta = \frac{2M\sigma^2(1-p)}{N^2}\sum_{j\geq 0}\alpha^2(j)$$

  Convergence rate of mean

$$\|\mathbb{E}[\mathbf{x}(i)] - r\mathbf{1}\| \leq \left(e^{-\lambda_2(\overline{L})\left(\sum_{0\leq j\leq i-1}\alpha(j)\right)}\right)\|\mathbb{E}[\mathbf{x}(0)] - r\mathbf{1}\|$$

Consensus: Quantized and Random Network

$$\mathbf{x}(i+1) = \mathbf{x}(i) - \alpha(i)\left[L(i)\mathbf{x}(i) + \Upsilon(i) + \Psi(i)\right]$$

- **Dither and quantization noise**
- **No distributional assumption, only finite 2^{nd} order moment**
- **Proof:**
  - **Characterize supremum over all sample paths of state of quantizer**
  - **Use maximal inequalities for submartinglae and supermartingale seq.**
  - **Derive prob. Bounds on excursions of sample paths**
  - **MMSE:**

$$\begin{aligned}\mathbb{E}\left[\left\|\mathbf{x}(i) - r\mathbf{1}\right\|^2\right] \;\leq\; & \frac{1}{\lambda_2(\overline{L})} e^{-\left(2\frac{\lambda_2^2(\overline{L})}{\lambda_N(\overline{L})} - \varepsilon\right)\sum_{j=i_\varepsilon}^{i-1}\alpha(j)} \mathbb{E}\left[\left\|\mathbf{x}_{\mathcal{C}^\perp}(i_\varepsilon)\right\|^2\right] + \frac{1}{\lambda_2(\overline{L})}\sum_{j=i_\varepsilon}^{i-1}\left[\; e^{-\left(2\frac{\lambda_2^2(\overline{L})}{\lambda_N(\overline{L})} - \varepsilon\right)\sum_{l=j+1}^{i-1}\alpha(l)} \right) g(j) \Bigg] \\ & + \frac{2|\mathcal{M}|\Delta^2}{3}\sum_{j=0}^{i-1}\alpha^2(j) \qquad (67)\end{aligned}$$

Distributed Nonlinear Estimation

$$\widetilde{\mathbf{x}}(i+1) = \widetilde{\mathbf{x}}(i) - \beta(i)\left(L(i) \otimes I_M\right)\widetilde{\mathbf{x}}(i) - \alpha(i)\left[\widetilde{\mathbf{x}}(i) - J(\mathbf{z}(i))\right] - \beta(i)\left(\boldsymbol{\Upsilon}(i) + \boldsymbol{\Psi}(i)\right)$$

$$\mathbf{x}(i) = \left[\left(h^{-1}(\widetilde{\mathbf{x}}_1(i))\right)^T \cdots \left(h^{-1}(\widetilde{\mathbf{x}}_N(i))\right)^T\right]^T$$

- **Nonlinear equivalent of observability condition**
- **Two time scales**

Conclusions

- High dimensional consensus:
  - Extends consensus
  - Large classes of distributed algorithms
- Random networks: lnk failures
- Noisy communications
- Errors in structural parameters
- Consensus with random links and noisy communications:
  - Stochastic approximation
- Consensus with quantized data and random links:
  - Stochastic approx not sufficient
  - Comparison argumanets
- Distributed estimation

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