PMUs Clock De-Synchronization Compensation for Smart Grid State Estimation – 2-nodes Toy Example

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EXPRESSION OF THE POSTERIOR VARIANCE FOR THE 2-NODES TEST-CASE

Let us consider the 2-nodes test-case where we assume M PMU measurements have been collected. The associated output matrix is given by

$$C = \begin{bmatrix} 1 & 1 & \frac{T}{M} \\ 1 & 1 & \frac{2T}{M} \\ & \vdots & \\ 1 & 1 & \frac{MT}{M} \end{bmatrix}$$

Let the prior covariance matrix be $P_0 = \text{diag}(\sigma_\theta^2, \sigma_\beta^2, \sigma_\alpha^2)$. Hence the posterior covariance, in information form is given by

$$P(\sigma_{\text{pmu},\theta}^2, \sigma_\theta^2, \sigma_\beta^2, \sigma_\alpha^2, M, T) = \left(P_0^{-1} + \frac{1}{\sigma_{\text{pmu},\theta}^2} C^T C\right)^{-1} = \frac{1}{\det(P^{-1})} \begin{bmatrix} \bar{p}_{11} & \bar{p}_{12} & \bar{p}_{13} \\ \bar{p}_{12} & \bar{p}_{22} & \bar{p}_{23} \\ \bar{p}_{13} & \bar{p}_{23} & \bar{p}_{33} \end{bmatrix}$$

which can be computed in closed form. First note that

$$C^T C = \begin{bmatrix} M & M & \frac{T}{M}\sum_{k=1}^{M} k \\ M & M & \frac{T}{M}\sum_{k=1}^{M} k \\ M & \frac{T}{M}\sum_{k=1}^{M} k & \frac{T^2}{M^2}\sum_{k=1}^{M} k^2 \end{bmatrix},$$

with $\sum_{k=1}^{M} k = \frac{M(M+1)}{2}$ and $\sum_{k=1}^{M} k^2 = \frac{M^3}{3} + \frac{M^2}{2} + \frac{M}{6}$. After some tedious algebraic manipulations we have

$$\begin{aligned}
\det(P^{-1}) &= \frac{1}{\sigma_\theta^2 \sigma_\beta^2 \sigma_{\text{pmu},\theta}^2} \left(\left(\left(\frac{M^2}{3} + \frac{M}{2} + \frac{1}{6} \right) \frac{1}{M} + (M^2 - 1) \frac{\sigma_\theta^2 + \sigma_\beta^2}{12 \sigma_{\text{pmu},\theta}^2} \right) T^2 + M \frac{\sigma_\theta^2 + \sigma_\beta^2}{\sigma_\alpha^2} + \frac{1}{\sigma_\alpha^2} \right), \\
\bar{p}_{11} &= \frac{1}{\sigma_{\text{pmu},\theta}^2} \left(\left(\frac{M^2}{3} + \frac{M}{2} + \frac{1}{6} \right) \frac{1}{M \sigma_\beta^2} + \frac{M^2 - 1}{12 \sigma_{\text{pmu},\theta}^2} \right) T^2 + \left(\frac{M}{\sigma_{\text{pmu},\theta}^2} + \frac{1}{\sigma_\beta^2} \right) \frac{1}{\sigma_\alpha^2}, \\
\bar{p}_{12} &= -\frac{1}{\sigma_{\text{pmu},\theta}^2} \left(\frac{M^2 - 1}{12 \sigma_{\text{pmu},\theta}^2} T^2 + \frac{M}{\sigma_\alpha^2} \right), \\
\bar{p}_{13} &= -\frac{M+1}{2 \sigma_\beta^2 \sigma_{\text{pmu},\theta}^2} T, \\
\bar{p}_{22} &= \frac{1}{\sigma_{\text{pmu},\theta}^2} \left(\left(\frac{M^2}{3} + \frac{M}{2} + \frac{1}{6} \right) \frac{1}{M \sigma_\theta^2} + \frac{M^2 - 1}{12 \sigma_{\text{pmu},\theta}^2} \right) T^2 + \left(\frac{M}{\sigma_{\text{pmu},\theta}^2} + \frac{1}{\sigma_\theta^2} \right) \frac{1}{\sigma_\alpha^2}, \\
\bar{p}_{23} &= -\frac{M+1}{2 \sigma_\theta^2 \sigma_{\text{pmu},\theta}^2} T, \\
\bar{p}_{33} &= \frac{1}{\sigma_\theta^2 \sigma_\beta^2} \left(1 + \frac{\sigma_\theta^2 + \sigma_\beta^2}{\sigma_{\text{pmu},\theta}^2} M \right).
\end{aligned}$$

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Thanks to the expressions above it is possible to perform a limit study for $MT \to \infty$. In particular considering the dominant terms in each one of the $\bar{p}_{ij}$ we have

$$\begin{aligned}
\det(P^{-1}) &\to \frac{\sigma_\theta^2 + \sigma_\beta^2}{12\sigma_\theta^2\sigma_\beta^2(\sigma_{\text{pmu},\theta}^2)^2} M^2T^2\,, \\
\bar{p}_{11} &\to \frac{1}{12(\sigma_{\text{pmu},\theta}^2)^2} M^2T^2\,, \\
\bar{p}_{12} &\to -\frac{1}{12(\sigma_{\text{pmu},\theta}^2)^2} M^2T^2\,, \\
\bar{p}_{13} &\to -\frac{1}{2\sigma_\beta^2\sigma_{\text{pmu},\theta}^2} MT\,, \\
\bar{p}_{22} &\to \frac{1}{12(\sigma_{\text{pmu},\theta}^2)^2} M^2T^2\,, \\
\bar{p}_{23} &\to -\frac{1}{2\sigma_\theta^2\sigma_{\text{pmu},\theta}^2} MT\,, \\
\bar{p}_{33} &\to \frac{\sigma_\theta^2 + \sigma_\beta^2}{\sigma_\theta^2\sigma_\beta^2\sigma_{\text{pmu},\theta}^2} M\,.
\end{aligned}$$

Thus, for the entire matrix, we have

$$P(\sigma_{\text{pmu},\theta}^2, \sigma_\theta^2, \sigma_\beta^2, \sigma_\alpha^2, M, T) \to \begin{bmatrix} \frac{\sigma_\theta^2\sigma_\beta^2}{\sigma_\theta^2+\sigma_\beta^2} & -\frac{\sigma_\theta^2\sigma_\beta^2}{\sigma_\theta^2+\sigma_\beta^2} & -\frac{6\sigma_\theta^2\sigma_{\text{pmu},\theta}^2}{\sigma_\theta^2+\sigma_\beta^2}\frac{1}{MT} \\ -\frac{\sigma_\theta^2\sigma_\beta^2}{\sigma_\theta^2+\sigma_\beta^2} & \frac{\sigma_\theta^2\sigma_\beta^2}{\sigma_\theta^2+\sigma_\beta^2} & -\frac{6\sigma_\beta^2\sigma_{\text{pmu},\theta}^2}{\sigma_\theta^2+\sigma_\beta^2}\frac{1}{MT} \\ -\frac{6\sigma_\theta^2\sigma_{\text{pmu},\theta}^2}{\sigma_\theta^2+\sigma_\beta^2}\frac{1}{MT} & -\frac{6\sigma_\beta^2\sigma_{\text{pmu},\theta}^2}{\sigma_\theta^2+\sigma_\beta^2}\frac{1}{MT} & \frac{12\sigma_{\text{pmu},\theta}^2}{MT^2} \end{bmatrix}$$

and, in particular,

$$\lim_{MT\to\infty} P(\sigma_{\text{pmu},\theta}^2, \sigma_\theta^2, \sigma_\beta^2, \sigma_\alpha^2, M, T) = \begin{bmatrix} \frac{\sigma_\theta^2\sigma_\beta^2}{\sigma_\theta^2+\sigma_\beta^2} & -\frac{\sigma_\theta^2\sigma_\beta^2}{\sigma_\theta^2+\sigma_\beta^2} & 0 \\ -\frac{\sigma_\theta^2\sigma_\beta^2}{\sigma_\theta^2+\sigma_\beta^2} & \frac{\sigma_\theta^2\sigma_\beta^2}{\sigma_\theta^2+\sigma_\beta^2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

It is interesting to observe that the uncertainty on β and θ does not go to zero. This is due to the fact they are linearly dependent variables (as can be seen from the output matrix C). Moreover, the nonzero elements of P form a 2×2 rank-1 matrix meaning that the uncertainties of offset and phase live in a one dimensional subspace and that, having perfect knowledge of one parameter directly translates in the perfect reconstruction of the remaining one. Final note regards p_{33}. It can be seen that it converges to zero faster than the other elements, specifically as $1/T^2$. Hence it is possible to express p_{33} as follows

$$p_{33}\left(\frac{1}{MT^2}\right) = \frac{\gamma}{MT}\frac{1}{T} + o\left(\frac{1}{T}\right)$$