

# Optimal Voltage Support and Stress Minimization in Power Networks

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**Abstract**—A standard operational requirement in power systems is that the voltage magnitudes must lie within pre-specified bounds. Conventional wisdom suggests that such a tightly regulated voltage profile should also guarantee a secure system, operating far from static bifurcation instabilities such as voltage collapse. Here we demonstrate that this conclusion is generally false, and that the distance from voltage collapse is a systems-level objective distinct from ensuring voltage limits. We formulate an optimization problem which maximizes the distance to voltage collapse through injections of reactive power, subject to power flow and operational voltage constraints. By exploiting a linear reformulation of the power flow equations we arrive at a convex reformulation which can be efficiently solved for the optimal reactive power injections. We illustrate the performance of our results with the IEEE30 bus network.

**Index Terms**—power networks, voltage support, reactive power compensation, sparsity-promoting optimization, optimal placement.

## I. INTRODUCTION

The widespread penetration of distributed renewable generation, characterized by high variability and fast dynamics, negatively impacts the voltage profile of a power network. Voltage controllers are therefore required to guarantee constraint satisfaction and safe operation of the network. Techniques for voltage support include shunt and static VAR compensation [1], [2], series compensation [3], off-nominal transformer tap ratios [4], synchronous condensers [5], and more inverters operating away from unity power factor [6], [7]. For a detailed description of the reactive power compensation technologies we refer the reader to [8].

Traditionally, the main purpose of voltage support is to maintain voltage magnitudes tightly within predetermined bounds (e.g., within 5% of some nominal level). Intuitively, such a tightly regulated voltage profile should also guarantee a large stability margin against static bifurcation instabilities such as voltage collapse. A key direction in power system stability analysis has been the development of indices quantifying a power network's proximity to voltage collapse. A broad overview of this large subfield can be found in [9], [10], [11], [12], [13]. The existing approaches are largely based on numerical methods. They often require either continuation power flow [14] to identify the insolvability

boundary, or repeated computation of loading margins in varying directions of parameter-space [15].

In this paper we combine voltage control with metrics quantifying the distance to voltage collapse in order to maximize the networks stability margin. While the usual objective in voltage support problems is the *security task* of confining voltage magnitudes within predetermined bounds, here we follow an alternative approach: we attempt to minimize a measure of the stress experienced by the network subject to operational constraints. This approach stems from [16], where a condition was introduced which quantifies the network stress measures and the proximity to voltage collapse. Based on this condition we pursue a novel system-level formulation of optimal voltage support encoded as an optimization problem with *stress-minimization*, i.e., maximization of the distance to voltage collapse, as objective and subject to voltage security constraints. Our decision variables are reactive power injections at a subset of buses equipped with voltage control equipment. This approach allows us to satisfy a *local* voltage constraint requirements while optimizing a *system-level* voltage stability margin. By exploiting linear approximation of the power flow equations, we convexify our problem which can then be efficiently solved for the optimal injections.

Compared to other approaches to voltage support problems, our results do not rely on the assumption of a radial (i.e., acyclic) power grid topology [6], [7]. This makes our approach appealing for both power transmission and distribution networks. Different from the reactive power compensation literature [17], [6], [18] and from the voltage support literature [7], we seek to maximize a voltage stability margin as opposed to related objectives such as minimizing power losses.

The remainder of this paper is organized as follows: in Section II, we recall the required preliminaries. In Section III we present the main contribution of the paper. Firstly, we review the typical objectives for voltage regulation problems and formulate our optimization problem. Secondly, we present and solve the convex reformulation of the problem. In Section IV we present some test cases. Finally, Section V offers concluding remarks.

## II. PRELIMINARIES

### A. Power Network, Generator and Load Models

A power network can be modelled as a connected, undirected and complex-weighted graph  $\mathcal{G}(\mathcal{V}, \mathcal{E})$  where  $\mathcal{V} = \{1, \dots, n\}$  ( $|\mathcal{V}| = n$ ) represents the set of nodes (or *buses*),  $\mathcal{E}$  ( $|\mathcal{E}| = m$ ) is the set of edges (or *branches*) connecting the nodes, that is the set of unordered pairs  $(h, k)$ ,  $h, k \in \mathcal{V}$ ,

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such that  $h$  and  $k$  are connected to each other. Under synchronous steady-state operating conditions, all the electric quantities are sinusoidal signals at the same frequency. At every bus  $h \in \mathcal{V}$  we have the following phasor quantities:

- nodal voltage:  $v_h = V_h \exp(j\theta_h) \in \mathbb{C}$ ;
- current injection:  $i_h = I_h \exp(j\psi_h) \in \mathbb{C}$ ;
- power injection:  $s_h = p_h + jq_h = v_h i_h^* \in \mathbb{C}$ .

where  $V_h, \theta_h, I_h, \psi_h, p_h, q_h \in \mathbb{R}$  and  $(\cdot)^*$  denotes the complex conjugate operator. A standard assumption on high-voltage networks is that transmission lines are dominantly inductive, and are therefore approximated as pure susceptances. For recent voltage control studies on lossy networks, see [6], [7], [18]. It is convenient to introduce the symmetric susceptance matrix  $B \in \mathbb{R}^{n \times n}$  which describes the weighted connections among the nodes of the graph. Its  $hk$ -th entry is defined as

$$B_{hk} = \begin{cases} 0, & \text{if } (h, k) \notin \mathcal{E}; \\ B_{hk} > 0, & \text{if } (h, k) \in \mathcal{E}, h \neq k; \\ -\sum_{h \neq k} B_{hk} + B_{h,\text{shunt}}, & \text{if } h = k; \end{cases}$$

where  $-B_{hk}$  is the line susceptance and  $B_{h,\text{shunt}} > 0$  (resp.  $B_{h,\text{shunt}} < 0$ ) represents the capacitive (resp. inductive) shunt susceptance to ground. The set of buses  $\mathcal{V}$  can be partitioned into the *loads* set,  $\mathcal{V}_L$ , and the *generators* set,  $\mathcal{V}_G$ , i.e.,  $\mathcal{V} = \mathcal{V}_L \cup \mathcal{V}_G$ , where we assume  $|\mathcal{V}_L| = n_\ell$  and  $|\mathcal{V}_G| = n_g$ . After potentially relabelling the buses, we partition the susceptance matrix according to loads and generators as

$$B = \begin{pmatrix} B_{LL} & B_{LG} \\ B_{GL} & B_{GG} \end{pmatrix}. \quad (1)$$

We refer to  $B_{LL}$  as the *grounded susceptance matrix*.

*Assumption 1 (Properties of  $B_{LL}$ ):*

- $-B_{LL}$  is a non-singular symmetric M-matrix<sup>1</sup>;
- the graph associated to the  $B_{LL}$  matrix (i.e., the graph induced by the load buses  $\mathcal{V}_L$ ) is connected.

Assumption 1 (i) is typically verified in practice [19], and always satisfied in the absence of line-charging and shunt capacitors. Assumption 1 (ii) can be made without loss of generality, since connected components of the induced graph will be electrically isolated from one another by generator buses. By collecting all voltages, currents and powers into vectors  $v, i, s \in \mathbb{C}^n$ , Kirchhoff's and Ohm's laws lead to

$$i = jBv, \quad (2)$$

where  $j$  denotes the imaginary unit. From (2) we can write the *Power Flow Equations* (PFEs) as

$$s = [v]i^* = [v](jBv)^*, \quad (3)$$

where  $[\cdot]$  denotes the diagonalization operator. Expanding (3), for each  $h \in \mathcal{V}$  the real and imaginary parts must satisfy

$$p_h(v) = \sum_{k=1}^n B_{hk} V_h V_k \sin(\theta_h - \theta_k), \quad (4a)$$

$$q_h(v) = -\sum_{k=1}^n B_{hk} V_h V_k \cos(\theta_h - \theta_k). \quad (4b)$$

The PFEs (4a)–(4b) describe the relations among the buses through the branches, while the behavior of each bus is specified by the particular model assumed for generators and loads. In this paper generators are modelled as standard *PV* buses [20], i.e., node  $h \in \mathcal{V}_G$  is specified by a fixed voltage magnitude  $V_h$  and a fixed active power generation level  $P_h$ . Instead, loads are modelled as *PQ* buses [21], [22], namely, for  $h \in \mathcal{V}_L$  active and reactive power injections  $P_h$  and  $Q_h$  are assumed fixed. In our setup, this *PQ* model refers to not only loads, but also to sources interfaced with power electronics and voltage support equipment such as synchronous condensers. While our results extend to ZIP load models [20], for simplicity of presentation we restrict ourselves to constant power loads  $Q_h(V_h) = Q_h$ .

### B. Decoupled Reactive Power Flow & Critical Load Matrix

Under normal operating conditions, the high-voltage operating point is characterized by small voltages angle differences [21], which we formalize in the following assumption.

*Assumption 2 (Decoupling Assumption):* In steady state operating conditions it holds that

$$|\theta_h - \theta_k| = 0 \quad \forall h, k : (h, k) \in \mathcal{E}.$$

Under Assumption 2 the Reactive Power Flow Equations (RPFEs) (4b) simplifies in vector notation to

$$q(V) = -[V]BV. \quad (5)$$

We now take into account the models and the partition introduced in Sections II-A and, accordingly we partition the vectors of voltage magnitudes and reactive power injections as

$$V = \begin{bmatrix} V_L \\ V_G \end{bmatrix}, \quad Q = \begin{bmatrix} Q_L \\ Q_G \end{bmatrix}.$$

Combining the power flow (5), the loads model and the partitioning (1), the power balance  $Q_h = q_h(V)$  at each load  $h \in \mathcal{V}_L$  can be written in vector notation as

$$Q_L = -[V_L] (B_{LL}V_L + B_{LG}V_G), \quad (6)$$

or even more simply

$$Q_L = -[V_L]B_{LL}(V_L - V_L^*). \quad (7)$$

Here,  $V_L^*$  is the high-voltage open-circuit solution of (6) without loading:  $Q_L = \mathbb{0}_{n_\ell}$ . We refer to it as the *open-circuit profile*. Since  $V_L^*$  corresponds to the voltage profile for zero loading, it could be referred to as the *zero-load profile* as well. Under Assumption 1, the open-circuit profile  $V_L^*$

$$V_L^* := -B_{LL}^{-1}B_{LG}V_G \quad (8)$$

is always well defined. Once (7) is solved for an operating point  $V_L$ , the reactive power injections at generators buses  $\mathcal{V}_G$  are uniquely determined by substituting the operating point into the final  $m$  equations in (5). It is useful to recall the definition of the *critical load matrix* from [16] which quantifies the stiffness of the transmission network

<sup>1</sup>An M-matrix  $A$  is a matrix with negative off-diagonal elements and positive diagonal ones which can be expressed in the form  $A = sI - B$ , with  $b_{ij} \leq 0$ ,  $s > \rho(B)$ , where  $\rho(B)$  is the maximum of the moduli of the eigenvalues of  $B$  and  $I$  is the identity matrix.

by combining the network structure, generator voltages, and generator locations into one matrix.

*Definition 3 (Critical Load Matrix):* Given the grounded susceptance matrix  $B_{LL}$  and the open-circuit profile  $V_L^*$  as defined in (8), the critical load matrix  $Q_{\text{crit}}$  is defined as

$$Q_{\text{crit}} := \frac{1}{4}[V_L^*]B_{LL}[V_L^*]. \quad (9)$$

### C. Normalized Coordinate System

It is convenient to represent (7) using a normalized set of variables. Using the open-circuit voltages  $V_L^*$  defined in (8), we denote the vector of percentage voltage deviations as

$$v := [V_L^*]^{-1}V_L. \quad (10)$$

Note that if the open-circuit profile is flat ( $V_L^* = \alpha \mathbb{1}$  for some  $\alpha > 0$ ), then  $v$  is simply the standard per unit voltage. In general, however, due to inhomogeneous generators voltage set points and the presence of shunt compensation,  $V_L^*$  is not flat and the scalings in (10) are non-uniform.

From Definition 3, it is possible to rewrite the RPFs (7) in the normalized coordinates (10) as

$$Q_L = [v]Q_{\text{crit}}(v - \mathbb{1}). \quad (11)$$

## III. PROBLEM FORMULATION

In this section we present our novel problem formulation which we refer to as the *Stress Minimization Problem*.

### A. Security Constraints

A common operational requirement is that the load bus voltage magnitudes must lie within a predefined percentage deviation from a reference voltage. This tight clustering of voltages is due to the following reasons:

- i) The loads are designed to operate with a voltage in a narrow region around the network base voltage;
- ii) A flat voltage profile minimizes the reactive flows and, consequently, minimizes the total power losses;
- iii) A flat voltage profile near the nominal voltage minimizes the operational wear on system components, which are designed to work at the nominal voltage level;
- iv) Intuitively, a flat voltage profile indicates that the network is safe from voltage collapse. We will in fact show that this heuristic does not generally ensure a good margin against static voltage instability.

We formalize this requirement by defining the *secure set*.

*Definition 4 (Secure set):* Given a reference voltage  $V_N \in \mathbb{R}_{>0}$  and a percentage deviation  $\alpha > 0$ , the *secure set*  $\mathbb{V}$  is defined as

$$\mathbb{V} := \left\{ V_L \in \mathbb{R}^{n_\ell} \mid \frac{\|V_L - V_N \mathbb{1}\|_\infty}{V_N} \leq \alpha \right\},$$

or, equivalently

$$\mathbb{V} := \left\{ v \in \mathbb{R}^{n_\ell} \mid \frac{\|[V_L^*]v - V_N \mathbb{1}\|_\infty}{V_N} \leq \alpha \right\}. \quad (12)$$

The common local “security” requirement is that the vector of percentage voltage deviations defined in (10) lies within the secure set, i.e.,

$$v \in \mathbb{V}. \quad (13)$$

While this represents a baseline operational requirement, under some circumstances it may not be sufficient to ensure safe grid operation. We present a simple example in which voltages remain within operational bounds, but the operating point is extremely sensitive to changes in load demands.

*Example 5 (Security requirement inadequacy):* Consider the simple two-buses case study consisting of a PV bus source connected to a PQ load and a collocated shunt capacitor, as illustrated in Figure 1a. Figures 1b–1c show the QV *nose curve* plot where:

- the blue curves represent the voltage solutions (high and low) of the RPFs (7) corresponding to different load conditions;
- the black horizontal dashed lines identify the secure set  $\mathbb{V}$ ;
- the orange vertical dashed lines identify the feasible loading limits induced by the secure set;
- the magenta dash-dotted line identifies the tangent line to the blue curve in the middle point of the orange lines, i.e., the sensitivity of the voltage w.r.t. load changes.

From Figure 1b, note that if the normalized load demand  $Q_L/Q_{\text{crit}}$  is too large, the operating point does not lie within  $\mathbb{V}$ . A standard policy is then to support the voltage level by adjusting the shunt compensation, i.e., by increasing  $B_{\text{shunt}}$ . We consider two different scenarios: Figure 1b shows the QV curve in the absence of shunt compensation ( $B_{\text{shunt}} = 0$ ), while Figure 1c shows the QV curve with  $B_{\text{shunt}} = 2.4$  S. Figure 1b shows that the security requirement (13), guarantees a “safe” distance to collapse, represented by the nose of the blue curve. Moreover, the slope of the QV curve at the operating point (i.e., the sensitivity of the voltage to changes in load) is small, meaning that relatively big changes in the load condition do not translate into big voltage changes. Conversely, in Figure 1c the security requirement is “dangerously” close to the nose of the curve. Moreover, the requirement alone does not discriminate the operating points within  $\mathbb{V}$ : indeed, it does not consider their proximity to the collapse point while, there are points (the left-most) which are preferable. Finally, the sensitivity line is steeper meaning that small changes in the load cause relatively big changes in the voltage, which is an undesirable property.

The previous analysis highlights that voltage constraint requirements alone can be insufficient. Note that the most preferable operating point within  $\mathbb{V}$  is the left-most voltage solution, identified by the intersection of the left orange line and the top black one. This is the feasible point farthest from voltage collapse. Moreover, the left-most point on the blue curve represents the open-circuit  $V_L^*$  solution. Then, a simple intuition is that by minimizing the distance of the operating point from the open-circuit solution — constrained to the fact that the operating point must belong to  $\mathbb{V}$  — we

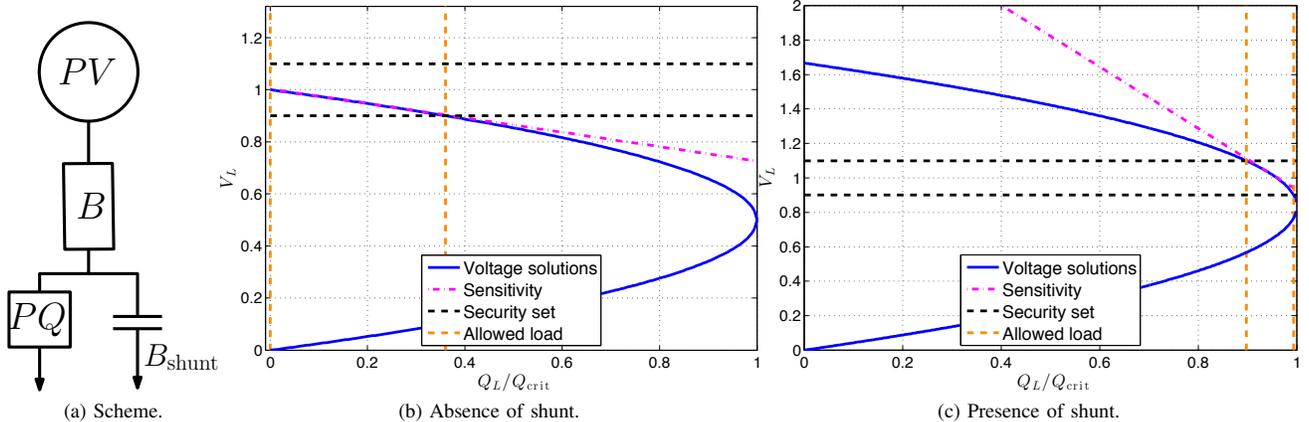


Fig. 1: Two-buses case: panel (a) plots the network scheme where  $V_N = 1$  [p.u.],  $B = 4$  S. Panels (b)–(c) plot the QV nose curve for two different configurations: (a) Absence of shunt capacitor,  $B_{\text{shunt}} = 0$ . (b) Presence of shunt capacitor,  $B_{\text{shunt}} = 2.4$  S. The blue curves represent the high and low voltage solutions of the RPFs. The black dashed horizontal lines identify the secure set. The orange dashed vertical lines are the allowed load which ensure the operating point to lie within the secure set. Finally, the magenta dash-dotted diagonal line represents the tangent line on the blue curve corresponding to the middle point of the orange lines.

will maximize the voltage collapse stability margin of the network.

### B. Network Stress Measure and Stress Minimization Problem

Based on the insights given by Example 5, we define the following measure for the network stress induced by the load.

*Definition 6 (Network Stress Measure):* Consider a power network  $\mathcal{G}(\mathcal{V}, \mathcal{E})$ , described by the RPFs (11), and let the open-circuit profile  $V_L^*$  be as in (8). Then, we define the network stress measure induced by the load as

$$J_{\text{stress}}(v) := \|v - 1\|_{\infty}, \quad (14)$$

where  $v = [V_L^*]^{-1}V_L$  is defined as in equation (10).

Definition 6 is based on the intuition that the open-circuit profile  $V_L^*$  represents the natural operating point in the absence of loading, i.e., the network is under “no stress”. In this sense, the stress function (14) quantifies the loading on the network conveniently expressed in the percentage deviation variable  $v$ .

In the following, we assume that a certain number of load buses can be equipped with additional controlled devices, e.g., synchronous condensers [5] or photovoltaic panels connected to the grid through power inverters. We assume these devices can provide a controllable amount of reactive power support, and in the following we model them as controllable sources of reactive power  $q$ , subject to upper and lower operational bounds. Specifically, the RPFs (11) are modified as

$$Q_L + q = [v]Q_{\text{crit}}(v - \mathbb{1}), \quad (15)$$

where  $q \in \mathbb{R}^{n_\ell}$  is such that  $q_{\min} \leq q \leq q_{\max}$ , being  $q_{\min}, q_{\max}$  the vectors of minimum and maximum prescribed reactive injections, respectively. The standing assumption is that, if a load bus  $h \in \mathcal{V}_L$  is not equipped with an additional

device, the corresponding entry  $q_h$  is fixed and equal to zero, i.e.,  $q_{\min, h} = q_{\max, h} = 0$ .

We now formulate our optimization problem of interest, which we refer to as the *Stress Minimization* problem.

*Problem 7 (Stress Minimization):* Consider a power network  $\mathcal{G}(\mathcal{V}, \mathcal{E})$  governed by the RPFs (15). Let  $J_{\text{stress}}(v)$  be defined as in (14). Then the goal is

$$\begin{aligned} & \underset{q \in \mathbb{R}^{n_\ell}}{\text{minimize}} && J_{\text{stress}}(v), \\ & \text{s.t.} && \begin{cases} v \in \mathbb{V}, \\ q_{\min} \leq q \leq q_{\max}. \end{cases} \end{aligned} \quad (16)$$

The main idea behind Problem 7 is that minimizing  $J_{\text{stress}}$  keeps the operating point of the network as far as possible from the tip of the nose curve, and thus far from voltage collapse. Voltage security requirements are incorporated into the problem as hard constraints.

Problem 7 is highly non-linear and non-convex, due to the quadratic behavior of (15). In the following, we convexify this problem through the use of a power flow linearization.

### C. Linear Approximation and Convexification

We now introduce a suitable linearization for Problem 7 which had been first presented in [23]. Assuming  $\|Q_L\| \sim 0$ , we expect the normalized normal profile (10) to be  $v \simeq \mathbb{1}$  (i.e.,  $V_L \simeq V_L^*$ ), which would be the exact high voltage solution corresponding to  $Q_L = 0$ . It follows by simple linearization that, to first order, the solution of the RPFs (15) is given by

$$\hat{v} = \mathbb{1} - \frac{1}{4}Q_{\text{crit}}^{-1}(Q_L + q). \quad (17)$$

More precisely: the first order solution of the RPFs is composed by the uniform profile  $\mathbb{1}$  plus a deviation which is a linear function of the network load profile. Using the

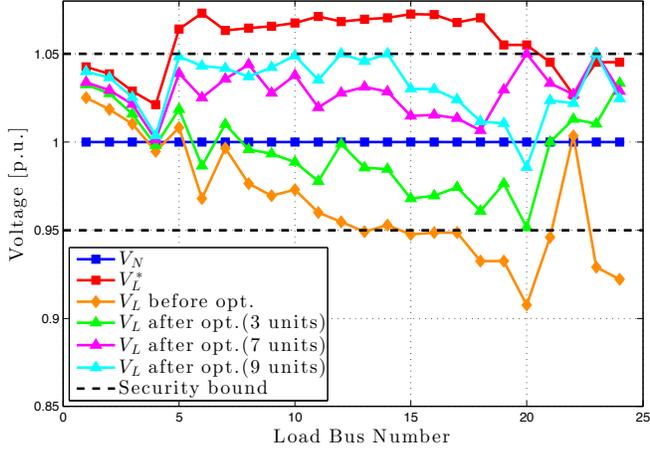


Fig. 2: Voltage profiles before and after the optimization.

linearization (17), the stress cost (14) can be approximated by

$$J_{\text{stress}}(v) = \|v - \mathbb{1}\|_{\infty} \propto \|Q_{\text{crit}}^{-1}(Q_L + q)\|_{\infty}. \quad (18)$$

Note that the approximated cost function is convex in the reactive power injections  $q$ . Exploiting the same linearization, the secure set  $\mathbb{V}$  is accordingly modified as

$$\widehat{\mathbb{V}} := \left\{ \widehat{v} \in \mathbb{R}^{n_{\ell}} \mid \frac{\| [V_L^*] \widehat{v} - V_N \mathbb{1} \|_{\infty}}{V_N} \leq \alpha \right\}.$$

Plugging the expression of  $\widehat{v}$  in (17) into the requirements in the definition of  $\widehat{\mathbb{V}}$ , we translate these voltage bounds into explicit bounds on our decision variables, the controlled reactive power injections  $q$ . After some algebraic manipulations, one finds that  $\widehat{v} \in \widehat{\mathbb{V}}$  if and only if

$$V_N(1 - \alpha)[V_L^*]^{-1} \mathbb{1} \leq \frac{1}{4} Q_{\text{crit}}^{-1}(Q_L + q) \leq V_N(1 + \alpha)[V_L^*]^{-1} \mathbb{1}.$$

These inequalities can be further rearranged to obtain

$$\xi_{\text{low}} \leq -Q_{\text{crit}}^{-1} q \leq \xi_{\text{up}},$$

where

$$\xi_{\text{low}} := 4(V_N(1 - \alpha)[V_L^*]^{-1} - \mathbb{1}) + Q_{\text{crit}}^{-1} Q_L, \quad (19a)$$

$$\xi_{\text{up}} := 4(V_N(1 + \alpha)[V_L^*]^{-1} - \mathbb{1}) + Q_{\text{crit}}^{-1} Q_L. \quad (19b)$$

We are now ready to present the convexified version of Problem 7.

**Problem 8 (Convex Stress Minimization):** Consider a power network  $\mathcal{G}(\mathcal{V}, \mathcal{E})$ , governed by the RPFES (15). Let  $V_N$ ,  $\alpha$ ,  $Q_{\text{crit}}$ ,  $V_L^*$ ,  $q_{\text{min}}$  and  $q_{\text{max}}$  be as in the previous problem. Finally, define  $\xi_{\text{low}}$  and  $\xi_{\text{up}}$  as in (19a)–(19b), respectively. Then we seek to

$$\begin{aligned} & \underset{q \in \mathbb{R}^{n_{\ell}}}{\text{minimize}} && \|Q_{\text{crit}}^{-1}(Q_L + q)\|_{\infty}, && (20) \\ & \text{s.t.} && \begin{cases} \xi_{\text{low}} \leq -Q_{\text{crit}}^{-1} q \leq \xi_{\text{up}}, \\ q_{\text{min}} \leq q \leq q_{\text{max}}. \end{cases} \end{aligned}$$

Before	After (3 units)	After (7 units)	After (9 units)
0.559	0.410	0.238	0.263

TABLE I: Values of  $\|Q_{\text{crit}}^{-1}(Q_L + q)\|_{\infty}$  before and after the optimization.

*Remark 9 (On the stress measure):* The stress measure (18) is inspired by recent results [16] on the solvability of the decoupled reactive power flow equations (7). In [16] it has been shown that if

$$\|Q_{\text{crit}}^{-1} Q_L\|_{\infty} < 1,$$

then (7) has a unique high-voltage solution safe from voltage collapse; for  $\|Q_{\text{crit}}^{-1} Q_L\|_{\infty} \geq 1$  voltage collapse may occur. From (18), we therefore see that  $J_{\text{stress}}(v)$  quantifies the stress experienced by the network.  $\square$

Observe that in Problem 8 both the cost function and the constraints are now convex in the optimization variables. Indeed, the security constraints are linear in  $q$  and identify a polytope. Thanks to the linearization introduced in Section III-C, Problem 8 can be efficiently solved by means of convex optimization techniques.

#### IV. SIMULATIONS

We now present a case study to show the effectiveness of the optimization procedure proposed. All the simulations have been done in MATLAB using MATPOWER [24] and CVX [25] for the power flow computations and the convex optimization, respectively. The test-bed consists of:

- IEEE 30 bus transmission grid;
- a reference voltage  $V_N = 1$  [p.u.];
- a secure threshold  $\alpha = 5\%$ ;
- capacity constraints  $q_{\text{min}} = -0.5 \times \|Q_L\|_{\infty}$  and  $q_{\text{max}} = 0.5 \times \|Q_L\|_{\infty}$ .

In all simulations we plot the actual profile  $V_L$  normalized with respect to the network base voltage, consistent with the MATPOWER convention, instead of the normalized profile  $v$ .

Figure 2 shows the performance before and after the optimization for different number of controlled loads. In particular, a set of 3, 7 and 9 loads out of 24, whose position have been randomly chosen, have been equipped with voltage control equipment which can inject reactive power. The addition of more controllable devices tends to improve the load profile and lower the objective minimum. However, since the positions of the controllable devices are not optimized, more controllable devices does not necessarily lead to a lower objective value. This can be seen either from Figure 2 or directly inspecting the value of the cost function after the optimization procedure, reported in Table I together with its initial value. Indeed, the 9 randomly selected units underperform the 7 randomly selected units; the placement of controllable units is important. To avoid this behavior, a suitable, e.g., greedy, approach for resource allocation could be added in order to improve the algorithm. Moreover, it is worth noting from Figure 2 how stress minimization and

classical voltage compensation do not coincide. Indeed, the open-circuit profile  $V_L^*$ , in general, does not even belong to the feasible set.

## V. CONCLUSIONS AND FUTURE DIRECTIONS

We have presented a novel optimization formulation for the voltage support via reactive power compensation whose cost function encodes the stress experienced by the grid, while the standard local security requirements are imposed as constraints. By exploiting a linearization of reactive power flow, the objective function becomes linear and convex in the optimization variables and the optimization problem can be effectively solved.

As future research direction, we will investigate how to equip the algorithm with a suitable placement procedure in order to let the operator choose a desired trade-off between performance and optimal resource allocation solution. We will further investigate the effectiveness of the presented approach with respect to different optimization variables, like the shunts capacitors or series compensation techniques. Moreover, we would like to investigate how the stress minimization objective could be implemented using a real-time feedback control algorithm, rather than a priori allocation and planning. Finally, we are working on extending our approach to coupled active/reactive power flows including conductances.

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