Derivation of fundamental quantities for the stability and convergence analysis

Technical report of
Discrete-time control of parallel kinematic systems

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***Abstract*— This document is a technical attachment to [1] (to appear) and contains the derivation and analysis of some quantities which are related to the stability and rate of convergence of the system introduced in the main paper.**

I. Stability and Convergence analysis

In this section we give the definition of the stability time $\bar{\tau}_s(\mu)$, the optimal time $\bar{\tau}_o(\mu)$ and the convergence rate $\rho(\mu)$. In order to evaluate upper bounds for asymptotic stability and rate of convergence, we need to study the following function

$$g(\tau;\mu) := |1-\tau| + \mu\tau^2 = \begin{cases} 1-\tau+\mu\tau^2 =: g^-(\tau;\mu) & \tau < 1 \\ -1+\tau+\mu\tau^2 =: g^+(\tau;\mu) & \tau \geq 1 \end{cases}$$

We will study the function $g(\tau;\mu)$ in three different scenarios: $\mu \in [0, \frac{1}{2})$, $\mu \in [\frac{1}{2}, 1)$ and $\mu \geq 1$. We start by observing that

$$g(0;\mu) = 1, \; g(1;\mu) = \mu, \; \frac{dg^+}{d\mu} = 1 + 2\mu\tau > 0$$

and by defining the minimum of $g^-(\tau;\mu)$ and its minimizer w.r.t. τ as

$$\tau_p(\mu) = \arg\min_\tau g^-(\tau;\mu) \Leftrightarrow \frac{dg^-(\tau;\mu)}{d\tau} = 0 \Longrightarrow \tau_p(\mu) = \frac{1}{2\mu}$$

We now note that in the first scenario $\mu \in [0, \frac{1}{2})$, $\tau_p(\mu) \geq 1$ which implies that the function $g(\tau;\mu)$ is monotonically decreasing for $\tau \in [0,1]$ and monotonically increasing for $\tau > 1$.
In the second scenario $\mu \in [\frac{1}{2}, 1)$, $\tau_p(\mu) < 1$, therefore $g(\tau;\mu)$ is monotonically decreasing for $\tau \in [0, \tau_p(\mu)]$ and monotonically increasing for $\tau > \tau_p(\mu)$.
Finally note that for $\mu < 1$, $g(1;\tau) < 1$, therefore there exists a unique $\overline{\tau}_s(\mu)$ such that $g(\overline{\tau}_s(\mu);\mu) = g^+(\overline{\tau}_s(\mu);\mu) = 1$, while for $\mu > 1$, $g(1;\tau) > 1$, therefore there exists a unique $\overline{\tau}_s(\mu)$ such that $g(\overline{\tau}_s(\mu);\mu) = g^-(\overline{\tau}_s(\mu);\mu) = 1$. A pictorial representation of the three scenarios is shown in Fig. 1. We are now ready to compute the stability region and convergence rate.

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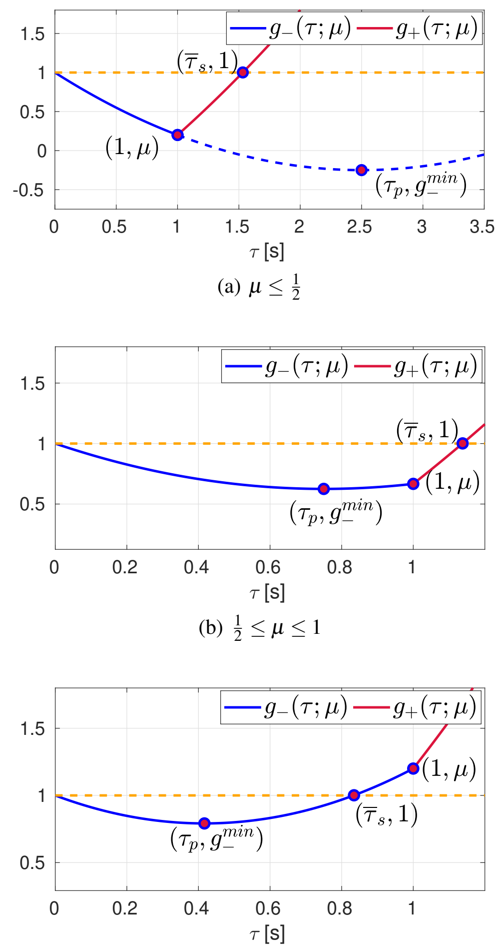

(a) $\mu \leq \frac{1}{2}$

(b) $\frac{1}{2} \leq \mu \leq 1$

(c) $\mu \geq 1$

Fig. 1: Representation of $g(\tau;\mu)$ in the three scenarios.

1) Stability ($g(\tau;\mu) < 1$): According to the analysis above, the stability set is given by:

$$\mathcal{T} := \{\tau \,|\, g(\tau;\mu) < 1\} = (0, \overline{\tau}_s(\mu))$$

More specifically, we have two scenarios depending whether the parameter μ is smaller or grater than unity.
If $\mu < 1$ then $-1+\tau+\mu\tau^2 = 1$. Hence:

$$\overline{\tau}_s(\mu) = \frac{-1+\sqrt{1+8\mu}}{2\mu} = \frac{4}{1+\sqrt{1+8\mu}}$$

while

$$\mu > 1 \Longrightarrow 1 - \tau + \mu\tau^2 = 1 \Longrightarrow \overline{\tau}_s(\mu) = \frac{1}{\mu}$$

which can be summarized in

$$\overline{\tau}_s(\mu) \begin{cases} \frac{4}{1+\sqrt{1+8\mu}} & \mu < 1 \\ \frac{1}{\mu} & \mu \geq 1 \end{cases} \quad (1)$$

2) Optimal gain and rate ($\min_\tau g(\tau;\mu)$*):* We now want to find the optimal stopping time $\overline{\tau}_o(\mu)$ in order to maximally decrease toward the origin, and the relative decrease rate $\rho(\mu)$, i.e.

$$\overline{\tau}_o(\mu) := \arg\min_\tau g(\tau;\mu), \quad \rho(\mu) = g(\overline{\tau}_o(\mu);\mu)$$

Once again, we can distinguish two scenarios, depending whether the parameter μ is smaller or greater than $\frac{1}{2}$
More specifically, for $\mu < \frac{1}{2}$ the function $g^-(\tau;\mu)$ is monotonically decreasing for $\tau < 1$, and therefore $\tau_o(\mu) = 1$, while for $\mu > \frac{1}{2}$ then $\tau_o(\mu) = \tau_p(\mu) = \frac{1}{2\mu}$. This can be summarized as

$$\overline{\tau}_o(\mu) = \begin{cases} 1 & \mu < \frac{1}{2} \\ \frac{1}{2\mu} & \mu \geq \frac{1}{2} \end{cases} \quad (2)$$

By substitution is easy to verify that

$$\rho(\mu) = \begin{cases} \mu & \mu < \frac{1}{2} \\ 1 - \frac{1}{4\mu} & \mu \geq \frac{1}{2} \end{cases} \quad (3)$$

REFERENCES

[1] E. Rossi, M. Tognon, R. Carli, L. Schenato, J. Cortés, and A. Franchi, “Cooperative aerial load transportation via sampled communication,” in *IEEE Control Systems Letters (L-CSS)*, 2019.