2013 European Control Conference (ECC)
July 17-19, 2013, Zürich, Switzerland.

A Game Theory Framework for Active Power Injection Management with Voltage Boundary in Smart Grids

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***Abstract*— Smart grids are a novel paradigm for energy distribution, where instead of the traditional directed flow from a producer to the consumers, several micro-generators are spread throughout the network. We focus on the problem of coordinating the injection of active power into the grid by the micro-generators. Each of them aims at injecting the maximum amount of power, satisfying some operative constraints such as voltage boundaries; a tradeoff must be found among these conflicting objectives. First, we characterize the active power increment region, i.e., the set of all the increments of injected power that, depending on the grid state, satisfy the voltage boundary. Based on this finding, we frame the problem within game theory and propose a distributed approach that achieves a fair share of the active power injection, while at the same time satisfying the voltage boundary.**

I. Introduction

The advent of distributed energy resources such as wind turbines, solar cells, or other renewable energy sources is deeply changing the actual power distribution scenario, and represents the key component of the *smart grid* [1]. These devices are called *micro-generators*, in contrast to traditional power plants, and are connected to the grid via electronic interfaces, called inverters (essentially, dc-ac converters) Their use can lead to a number of benefits for the electrical distribution system, such as reduction of line losses, voltage profile improvements, and decrease of emissions of pollutants [2]. The development of the smart grids is strictly related to the study of algorithms and methods to address the problems arising in the new energetic scenario, such as the bidirectionality of the power flow, the unpredictability of the energy supply from micro-generators and the management of active and reactive power flows [3], [4]. This last point is crucial, since the injection of power from the micro-generators can lead, if not conveniently regulated, to grid instability. For this reason, it is important to properly design the operation policy of the distributed generators (DGs). An approach to solve this problem is to study the injection region of the network [5]. Due to the possible large number of actors in the grid, distributed approaches are replacing traditional centralized control algorithms. If the system must satisfy voltage magnitude constraints, a popular distributed approach is the droop control of the inverters [6], [7], which does not require communications between the DGs. Instead, when the DGs have communication capabilities, more sophisticated algorithm can be developed [8], [9].

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We envision Game Theory, the popular branch of mathematics that studies the interactions and conflicts between multiple players within a common system [10], as a possible approach. In the present paper, the players are the microgenerators, also named *compensators* because they are the controllable part of the grid, that want to maximize the amount of active power that they inject into distribution network; the economic counterpart of this injection is that the injected power is sold to the main provider, and thus the microgenerators aim at increasing their revenues. Other works use game theory for similar goals involving control of the injected power. For example, in [11] and [12], the authors use a *minmax* approach, in which only one agent plays at each step, while the others maintain the injected power constant. In [13] the micro-generators try to construct the optimal *coalitions* that will exchange the power produced with the aim to minimize the power losses. In all the previous example, the compensators try to improve the global distribution system performance, and their goal functions are related to some performance index of the grid; in this paper, the compensators are only interested in maximizing a egoistic function, i.e. the amount of power that they inject.

II. Mathematical preliminaries and Notations

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \sigma, \tau)$ be a directed graph, where $\mathcal{V}$ is the set of nodes, $\mathcal{E}$ is the set of edges, with $n = |\mathcal{V}|, r = |\mathcal{E}|$. Moreover $\sigma, \tau : \mathcal{E} \to \mathcal{V}$ are two functions such that edge $e \in \mathcal{E}$ goes from the source node $\sigma(e)$ to the terminal node $\tau(e)$. Two edges e and e' are *consecutive* if the intersection $\{\sigma(e), \tau(e)\} \cap \{\sigma(e'), \tau(e')\}$ is not empty. A *path* is a sequence of consecutive edges (regardless of their direction).

If Γ is a $\mathbb{C}^{p\times q}$ matrix then $\Gamma(u, v)$ denotes the element in the u-th row and in the v-th column of Γ. Similarly, if $x \in \mathbb{C}^p$, its ith element is denoted by $x(i)$; also, $|x|$ is the vector whose elements are $|x|(i)=|x(i)|, \quad \forall i=1,\dots,p$, and x^2 is the vector whose elements are $x^2(i)=x(i)^2, \quad \forall i=1,\dots,p$. The operation of element-wise complex conjugation and matrix transposition are denoted by $^-$ and ^{T}, respectively. If $x, y \in \mathbb{R}^p$, $x \preceq y$ means that $x(i) \le y(i) \quad \forall i = 1,\dots,p$.

Let $A \in \{0, \pm 1\}^{r\times n}$ be the incidence matrix of the graph $\mathcal{G}$, defined via its elements

$$A(e, v) = \begin{cases} -1 & \text{if } v = \sigma(e) \\ 1 & \text{if } v = \tau(e) \\ 0 & \text{otherwise.} \end{cases}$$

If $\mathcal{W}$ is a subset of indices, we define as $\mathbf{1}_{\mathcal{W}}$ the column vector whose elements are

$$\mathbf{1}_{\mathcal{W}}(v)\begin{cases}1 & \text{if } v\in\mathcal{W}\\ 0 & \text{otherwise.}\end{cases}$$

Similarly, if w is an index, we denote by $\mathbf{1}_w$ the column vector whose value is 1 in position w, and 0 elsewhere, and we denote by $\mathbf{1}$ the column vector of all ones.

III. MICROGRID LINEAR MODEL

For the purpose of this paper, we model a microgrid as a directed graph $\mathcal{G}$, in which the edges in $\mathcal{E}$ represent the power lines, and the nodes in $\mathcal{V}$ represent both loads and generators that are connected to the microgrid. These include the residential and industrial consumers, microgenerators, and also the point of connection of the microgrid to the transmission grid (called point of common coupling, or PCC).

We limit our study to the steady state behavior of the system, where all voltages and currents are sinusoidal signals at the same pulsation ω_0. Each signal can therefore be represented via a complex number whose amplitude corresponds to the signal root-mean-square value, and whose phase corresponds to the phase of the signal at $t=0$. Therefore, y represents the signal $y(t)=|y|\sqrt{2}\sin(\omega_0 t+\angle y)$. The system state is described by the following system variables:

- $u\in\mathbb{C}^n$, where $u(v)$ is the grid voltage at node v;
- $i\in\mathbb{C}^n$, where $i(v)$ is the current injected by node v into the grid;
- $\xi\in\mathbb{C}^r$, where $\xi(e)$ is the current flowing on edge e.
- $s=p+iq\in\mathbb{C}^r$, where $s(v)$, $p(v)$ and $q(v)$ are the complex, the active and the reactive power injected by node v into the grid respectively;

The following equations (Kirchhoff's current and voltage law) are satisfied by u, i and ξ:

$$A^T\xi+i=0, \tag{1}$$
$$Au+Z\xi=0, \tag{2}$$

with Z=diag$(z(e),e\in\mathcal{E})$, and $z(e)\in\mathbb{C}$ being the impedance of power line e; thus, at each node v, the injected current $i(v)$ is related to voltage $u(v)$. We label the PCC as node 0 and take it as a constant voltage generator (slack bus), i.e.

$$u(0)=U_0\ . \tag{3}$$

We assume $U_0\in\mathbb{R}$. Instead, for every $v\in\mathcal{V}\setminus\{0\}$

$$u(v)\bar{i}(v)=s(v),\quad \forall v\in\mathcal{V}\setminus\{0\}, \tag{4}$$

Model (4) is called *constant power model*; each node will inject the nominal complex power $s(v)$ independently of its voltage $u(v)$. We define the complex-valued Laplacian as $\boldsymbol{L}=A^TZ^{-1}A$, and the *Green matrix* $\boldsymbol{X}$ as the only symmetric matrix that satisfies

$$\begin{cases}\boldsymbol{XL}=I-\mathbf{1}\mathbf{1}_0^T\\ \boldsymbol{X}\mathbf{1}_0=0,\end{cases} \tag{5}$$

Matrix $\boldsymbol{X}$ depends only on the topology of the microgrid power lines and their impedances. We denote $\boldsymbol{X}(i,j)=X(i,j)^{\angle\boldsymbol{X}(i,j)}$.

If we set the nominal voltage U_0 at the PCC, all the currents i and the voltages u are determined by

$$\begin{cases}u=\boldsymbol{X}i+U_0\mathbf{1}\\ \mathbf{1}^Ti=0\\ u(v)\bar{i}(v)=s(v),\quad\forall v\in\mathcal{V}\setminus\{0\},\end{cases} \tag{6}$$

where the first equation results from (1), (2), and (3).

Let s be the vector of all nominal complex powers $s(v)$, including $s(0):=-\sum_{v\in\mathcal{V}\setminus\{0\}}s(v)$. Then, from (6), a linear approximated model relating the voltages and the powers injected in the grid can be derived [14]

$$\begin{cases}u=\frac{\boldsymbol{X}}{U_0}\bar{s}+U_0\mathbf{1}\\ \mathbf{1}^Ts=0,\end{cases} \tag{7}$$

IV. ACTIVE POWER INJECTION

Let $\mathcal{C}\subset\mathcal{V}$ be the set of the compensators, and $m:=|\mathcal{C}|$ is its cardinality. For the sake of analytical simplicity, we also include the PCC in $\mathcal{C}$, even though it is not controllable, i.e., its power injection into the grid cannot be commanded. They represent the *smart* part of the grid. It is useful to distinguish the m components of u and p that represent the state of the compensators. Without loss of generality, assume they are at the first positions of the vectors, i.e.

$$u=\begin{bmatrix}u_{\mathcal{C}}\\ u_{\mathcal{V}\setminus\mathcal{C}}\end{bmatrix}\qquad p=\begin{bmatrix}p_{\mathcal{C}}\\ p_{\mathcal{V}\setminus\mathcal{C}}\end{bmatrix}$$

where $u_{\mathcal{C}}\in\mathbb{C}^m, p_{\mathcal{C}}\in\mathbb{R}^m$ and $u_{\mathcal{V}\setminus\mathcal{C}}\in\mathbb{C}^{n-m}, p_{\mathcal{V}\setminus\mathcal{C}}\in\mathbb{R}^{n-m}$. This induces the following partition on $\boldsymbol{X}$ as

$$\boldsymbol{X}=\begin{bmatrix}\boldsymbol{X}_{\mathcal{C}} & \boldsymbol{X}_{\mathcal{C},\mathcal{V}\setminus\mathcal{C}}\\ \boldsymbol{X}^T_{\mathcal{C},\mathcal{V}\setminus\mathcal{C}} & \boldsymbol{X}_{\mathcal{V}\setminus\mathcal{C}}\end{bmatrix}.$$

We assume that there is no limit on the generation capability of the microgenerators. The only constraints come from the voltage profile quality and the voltage values that are guaranteed to the customers. A compromise must be sought among the microgenerators on the amount of active power that they will inject in the distribution system, in order to satisfy their objectives to inject the maximum possible amounts of energy. Also, the operative constraints must be respected; the latter can be translated into voltage magnitude boundaries, i.e., it has to be guaranteed

$$||u|-U_0\mathbf{1}|\preceq\beta U_0\mathbf{1} \tag{8}$$

where β is a confidence margin, e.g. β=0.05. We assume that only the compensators measure their voltages and communicate with each other. The optimal policy is to seek for

$$||u_{\mathcal{C}}|-U_0\mathbf{1}|\preceq\beta U_0\mathbf{1}$$

For notational simplicity, from now on, we will drop the subscript from $u_{\mathcal{C}}$. Moreover, we relax the constraint to require

$$|u|-U_0\mathbf{1}\preceq\beta U_0\mathbf{1} \tag{9}$$

Indeed, as compensators have no limits on their generation capability and they desire to inject the maximum allowed active power, the minimum voltage bound would never be active. For the sake of simplicity, we do not consider other possible constraints such as loss limits on individual lines or thermal losses of the line. Now, assume the network state at some instant to be represented by voltages u and powers s, and that the compensators active power injection changes by $\Delta \in \mathbb{R}^m$. By (7), the new voltages u'_Δ will be

$$u'_\Delta = u + \frac{\boldsymbol{X}_\mathcal{C}}{U_0}\Delta \tag{10}$$

We define $\psi^M = ((1+\beta)U_0)^2\mathbf{1} - |u|^2$ and $\psi_\Delta = |u'|^2 - |u|^2$. The elements of ψ_Δ and ψ^M can be negative. If u' is a feasible point, i.e. it satisfies (9), it holds

$$\psi_\Delta \preceq \psi^M \tag{11}$$

We define δ_i^M as the maximum change, possibly negative, in the active power injected by i such that, after the injection changing $\Delta_i^M = \delta_i^M \mathbf{1}_i$, from (10) it holds

$$\begin{cases} |u'_{\Delta_i^M}(i)| \le (1+\beta)\ U_0 \quad \forall i \in \mathcal{C}, \\ \exists j \in \mathcal{C} : |u'_{\Delta_i^M}(j)| = (1+\beta) U_0 \end{cases} \tag{12}$$

Clearly $\psi_{\Delta_i^M} \preceq \psi^M$. Let $\theta_{ij} = \angle \boldsymbol{X}(i,j) - \angle u(i)$ be the difference between the phase of $\boldsymbol{X}(i,j)$ and the phase of the phasor $u(i)$. We make the following

Assumption 1: Let the phase be $0 \le \angle \boldsymbol{X}(i,j) \le 90°$, and let $-90° + \angle \boldsymbol{X}(i,j) \le \angle u(i) \le 90° + \angle \boldsymbol{X}(i,j)$ so that $\cos\theta_{ij} = \cos(\angle \boldsymbol{X}(i,j) - \angle u(i)) \ge 0\ \forall i,j \in \mathcal{C}$.

Since $\forall i,j \in \mathcal{C}$, after changing the injection by $\Delta_j = \delta_j \mathbf{1}_j$,

$$|u(i)'_{\Delta_j}(i)|^2 = |u(i)|^2 + \delta_j^2 \frac{X(i,j)^2}{U_0^2} + 2\delta_j \frac{X(i,j)}{U_0}|u(i)|\cos\theta_{ij} \tag{13}$$

then if $\delta_j > 0$, $|u(i)'|$ will increase, and conversely decrease only if δ_j is negative. As a consequence, given voltages $u_\mathcal{C}$, the quantities δp_i^M will be either all positive or all negative.

We now prove some useful lemmas.

Lemma 2: If $n \ge 1$, $a_i \in \mathbb{R}$, $\quad i = 1, \dots, n$, consider

$$\Gamma = \begin{bmatrix} a_1 - a_1^2 & -a_1 a_2 & \dots & -a_1 a_n \\ -a_2 a_1 & a_2 - a_2^2 & \dots & -a_2 a_n \\ \dots & \dots & \dots & \dots \\ -a_n a_1 & -a_n a_2 & \dots & a_n - a_n^2 \end{bmatrix}$$

Then

$$\det \Gamma = \prod_{i=1}^n a_i \big(1 - \sum_{i=1}^n a_i\big) \tag{14}$$

Proof: See the appendix. ■

Lemma 3: Let $n \ge 1$, $0 \le a_i \le 1$, $i = 1, \dots, n$, $\sum_{i=1}^n a_i \le 1$. Then, $\forall x_i \in \mathbb{R}$, $i = 1, \dots, n$ it holds

$$\big(\sum_{i=1}^n a_i x_i\big)^2 \le \sum_{i=1}^n a_i x_i^2 \tag{15}$$

Proof: To check (15) we have to show that

$$\sum_{i=1}^n a_i x_i^2 - \big(\sum_{i=1}^n a_i x_i\big)^2 \ge 0 \tag{16}$$

It is easy to verify that the left hand side of (16) is a quadratic form that can be rewritten as

$$\begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^T \begin{bmatrix} a_1 - a_1^2 & \dots & -a_1 a_n \\ \dots & \dots & \dots \\ -a_n a_1 & \dots & a_n - a_n^2 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \tag{17}$$

from Lemma 2 and from the hypothesis it trivially follows that it is also positive semidefinite, as every principal minor of the matrix has non-negative determinant. ■

Assume there are ℓ compensators that want to change their active power injection to share the active power generation.

Proposition 4: Let $\Delta_1 = \delta_1 \mathbf{1}_1, \dots, \Delta_\ell = \delta_\ell \mathbf{1}_\ell$ be a sequence of possible changes on the injection of active power, each of them leading to a feasible point that satisfies (9). Let $\lambda \in \mathbb{R}^\ell$, with all the $\lambda(i)$'s greater than or equal to zero and such that $\mathbf{1}^T\lambda = 1$. Then the convex combination

$$\Delta_\lambda = \lambda_1 \Delta_1 + \dots + \lambda_\ell \Delta_\ell \tag{18}$$

leads, with respect to the model (10), to a feasible point.

Proof: Let u be the voltages of the compensator at some instant. If we change the amount of injected active power by (18), we obtain $\forall i \in \mathcal{C}$

$$\begin{aligned} |u'(i)|^2 &= \left(|u(i)| + \sum_{j=1}^\ell \frac{X(i,j)}{U_0}\lambda_j \delta_j \cos\theta_{ij}\right)^2 \\ &+ \left(\sum_{j=1}^\ell \frac{X(i,j)}{U_0}\lambda_j \delta_j \sin\theta_{ij}\right)^2 \end{aligned}$$

and so

$$\begin{aligned} \psi_{\Delta_\lambda}(i) &= \sum_{j=1}^\ell \left(\frac{X(i,j)}{U_0}\lambda_j \delta_j\right)^2 \\ &+ 2\sum_{j,k=1,k\neq j}^\ell \frac{X(i,j)X(i,k)}{U_0^2}\lambda_j\lambda_k\delta_j\delta_k\cos(\theta_{ij}-\theta_{ik}) \\ &+ 2\sum_{j=1}^\ell \frac{X(i,j)}{U_0}\lambda_j\delta_j\cos(\theta_{ij})|u(i)| \end{aligned} \tag{19}$$

From (13), we have

$$\begin{aligned} \sum_{j=1}^\ell \lambda_j \psi_{\Delta_j}(i) = &\sum_{j=1}^\ell \lambda_j \left(\frac{X(i,j)}{U_0}\delta_j\right)^2 \\ &+ 2\sum_{j=1}^\ell \lambda_j \frac{X(i,j)}{U_0}\delta_j\cos(\theta_{ij})|u(i)| \end{aligned} \tag{20}$$

Using (19) and (20), we get

$$\begin{aligned} \sum_{j=1}^\ell \lambda_j \psi_{\Delta_j}(i) - \psi_{\Delta_\lambda}(i) \ge &\sum_{j=1}^\ell \lambda_j \left(\frac{X(i,j)}{U_0}\delta_j\right)^2 \\ &- \left(\sum_{j=1}^\ell \lambda_j \frac{X(i,j)}{U_0}\delta_j\right)^2 \ge 0 \end{aligned}$$

and then, from Lemma 3 and from (11), it follows that

$$\psi_{\Delta_\lambda}(i) \le \sum_{j=1}^\ell \lambda_j \psi_{\Delta_j}(i) \le \psi^M(i)\sum_{j=1}^\ell \lambda_j = \psi^M(i)$$

which means that u' is feasible. ■

The previous Proposition 4 can be used to construct an m-dimensional region of the active powers injected by the compensators that is strictly contained in the region of feasible states of the distribution network, with respect to (9). The key is to compute the δ_i's exploiting the voltages phasorial measurements taken by phasor measurement unit (PMU) [15]. This could be done by a central control unit that receives the voltage measurements by the compensators and exploits the knowledge of the grid topology, or in a distributed iterative way.

V. Active Power Injection Game

Given any state of the grid, the compensators either can or cannot increase the amount of active power injected, depending on whether (9) is satisfied or not. The ℓ compensators wanting to change their injected active power need to reach a (possibly fair) agreement. To this end, we propose to employ a repeated games framework. The game will choose the active power change between the convex combinations of the Δ_i^M's. This is, from the analysis of Section IV, within the Pareto boundary. Moreover, if the lines can be regarded as resistive and the voltages phases are sufficiently small, in (13), we have $\cos\theta_{ij} \simeq \cos(\angle \boldsymbol{X}(i,j)) \simeq 1$, and so, if the active power injected changes by $\Delta \in \mathbb{R}^m$, $\forall i = 1,\ldots,m$

$$|u(i)'_\Delta| \simeq |u(i)| + \sum_{j=1}^{\ell} \Delta(j) \frac{X(i,j)}{U_0} \cos(\angle \boldsymbol{X}_{ij}) \qquad (21)$$

i.e. the voltages magnitudes are almost linear in the active power. Thus, if each δ_i^M leads the same compensator to have a voltage magnitude of $(1+\beta)U_0$, the Pareto boundary of the feasible region induced by the linear system (21) is the convex combination of the Δ_i^M's.

Now, we analyze *three different games* that can be used to decide the amount of the injected active power change. We assume that the number of playing compensators $\ell = 2^\alpha$ is a power of 2, and that the PCC does not take part in the game. This assumption is not restrictive since we could always add b dummy compensators with $\delta = 0$ such that $\ell + b = 2^\alpha$. At first, each compensator receives or computes the value $\delta_i^0 = \delta_i^M$. Then, the compensators are divided into pairs playing the following game. Let $i, j \in \mathcal{C}$ be the compensators that form one of these couples. They play the game

$$\max_{0\le\lambda\le1} \varphi(\lambda\delta_i^0, (1-\lambda)\delta_j^0) \qquad (22)$$

i.e., maximize a function that models the fairness of their agreement moving on the Pareto boundary approximation, obtaining a first stage game where

$$\delta_i^1 = \lambda_i^1\delta_i^M = \lambda\delta_i^M, \quad \delta_j^1 = \lambda_j^1\delta_j^M = (1-\lambda)\delta_j^M \qquad (23)$$

After this, each pair elects a representative, which will be paired with the representative of another pair. For instance, let the representatives of (i,j) and (f,g) be i and g respectively. Then, they play a second stage where

$$\max_{0\le\lambda\le1} \varphi(\lambda\delta_1^i, (1-\lambda)\delta_1^g) \qquad (24)$$

where φ is a proper function discussed later, obtaining

$$\delta_i^2 = \lambda_i^2\delta_i^1 = \lambda\delta_i^1 = \lambda_i^2\lambda_i^1\delta_i^M$$
$$\delta_g^2 = \lambda_g^2\delta_g^1 = (1-\lambda)\delta_g^1 = \lambda_g^2\lambda_g^1\delta_g^M$$

Now, i and g notify λ_i^2 and λ_g^2 to the node they were originally paired with (j and f, respectively), which compute

$$\delta_j^2 = \lambda_i^2\delta_j^1 = \lambda_i^2\lambda_j^1\delta_j^M, \quad \delta_f^2 = \lambda_g^2\delta_f^1 = \lambda_g^2\lambda_f^1\delta_f^M$$

This process is further iterated up to α times.

A suitable φ is the square of the geometric average, leading to a solution akin to Nash bargaining solution [16],

$$\varphi(x,y) = xy\ . \qquad (25)$$

It is easy to see that the solution of the k-th stage (played for example by i and j) using (25) is

$$\lambda_i^k = \frac{1}{2}, \quad \lambda_j^k = \frac{1}{2} \qquad (26)$$

and that the algorithm induces

$$\delta_1 = \frac{\delta_1^M}{2^\alpha} = \frac{\delta_1^M}{\ell}, \ldots, \delta_\ell = \frac{\delta_\ell^M}{2^\alpha} = \frac{\delta_\ell^M}{\ell} \qquad (27)$$

and so it can be computed just knowing the number of playing compensators in the microgrid (that would not be constrained to a power of 2) and the values $\delta_1^M, \ldots, \delta_\ell^M$.

Another function that can be used is

$$\varphi(x,y) = -\left|\frac{x}{\zeta_x} - \frac{y}{\zeta_y}\right| \qquad (28)$$

which trivially forces a solution characterized by the fact that the ratio δ_i/ζ_i is equal for each compensators i. Notice that, if the ζ_i's are chosen proportionally to the nominal generation capabilities of the inverters, the solution is similar to the one obtained by the classical droop control. It is easy to see that the solution of the k-th stage (played for example by i and j) using (28) is

$$\lambda_i^k = \frac{\zeta_i\delta_j^{k-1}}{\zeta_j\delta_i^{k-1} + \zeta_i\delta_j^{k-1}}, \quad \lambda_j^k = \frac{\zeta_j\delta_i^{k-1}}{\zeta_j\delta_i^{k-1} + \zeta_i\delta_j^{k-1}} \qquad (29)$$

Instead, the choice of all equal ζ_i's compels the compensators to make the same change of injected active power, ending in a solution

$$\delta_1 = \cdots = \delta_\ell \triangleq \delta_{eq} \qquad (30)$$

where δ_{eq} is the common value of all δ_i's.

As will be seen in Section VI, values $\delta_1^M, \ldots, \delta_\ell^M$ can be very different, and are heavily dependent by the compensators position in the grid. So, solution (27), in spite of its apparent fairness, can actually lead to very unbalanced outcomes, which means that some compensators are privileged because of their locations. On the other hand, the solution of (30) seems the most egalitarian, being all the δ_i's equal. Yet, if we compute the total injected active power $\sum_{i=1}^{\ell}\delta_i = \ell\delta_{eq}$, it can be much smaller than what computed as per (27), i.e., $\frac{1}{\ell}\sum_{i=1}^{\ell}\delta_i^M$. A possible trade-off is to move on the Pareto boundary between the solutions induced by

TABLE I
INITIAL ALLOCATION

node	$\|u\|$ [V]	δ^M [kW]
2	4742	7478
4	4744	3190
14	4708	2121
16	4723	4909
20	4722	2603
24	4688	2881
30	4673	1614
33	4647	1624

(26) and (29), i.e. at the k-th stage played by i and j, to choose

$$\begin{aligned} \lambda_i^k =& \frac{\delta_j^{k-1}}{\delta_i^{k-1}+\delta_j^{k-1}} + \eta\Big(\frac{1}{2} - \frac{\delta_j^{k-1}}{\delta_i^{k-1}+\delta_j^{k-1}}\Big) \\ \lambda_j^k =& \frac{\delta_i^{k-1}}{\delta_i^{k-1}+\delta_j^{k-1}} + \eta\Big(\frac{1}{2} - \frac{\delta_i^{k-1}}{\delta_i^{k-1}+\delta_j^{k-1}}\Big) \end{aligned} \tag{31}$$

with $0 \le \eta \le 1$, η being a parameter that has to be properly designed to obtain the desired solution.

VI. NUMERICAL RESULTS

As a low voltage testbed is currently missing in the literature, we considered a 4.8 kV testbed inspired from the standard test feeder IEEE37 [17]. However, we assume that the loads are balanced, and therefore all the signals can be described in a single-phase phasorial notation. As shown in Fig. 1, some of the nodes are microgenerators connected to the microgrid via power inverters. Following the model proposed in Section III, we consider every node (but the PCC) behaving as a constant-power device. At first

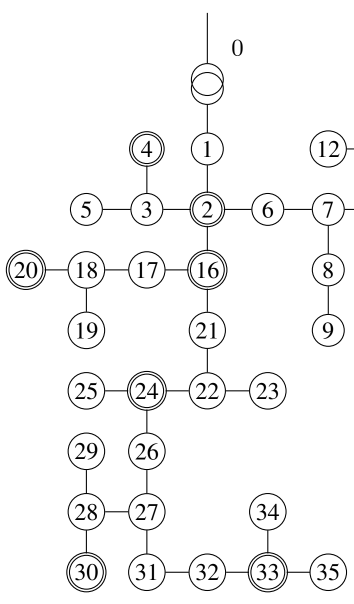

Fig. 1. Graph of a microgrid based on the IEEE37 test feeder [17]. The nodes represent microgenerators (circled), loads, and the PCC (node 0).

the ℓ playing compensators obtain or compute $\delta_1^M, \dots, \delta_\ell^M$. This computation is done considering constraint (9), with $\beta = 0.05$. The results are reported in Table I. After that,

TABLE II
RESULTING ALLOCATION BY USING (26)

node	λ	δ [kW]	$\|u'\|$ [V]
2	0.125	934.81	4873
4	0.125	398.75	4896
14	0.125	265.12	4870
16	0.125	613.68	4897
20	0.125	325.43	4914
24	0.125	360.18	4906
30	0.125	201.81	4896
33	0.125	203.00	4899

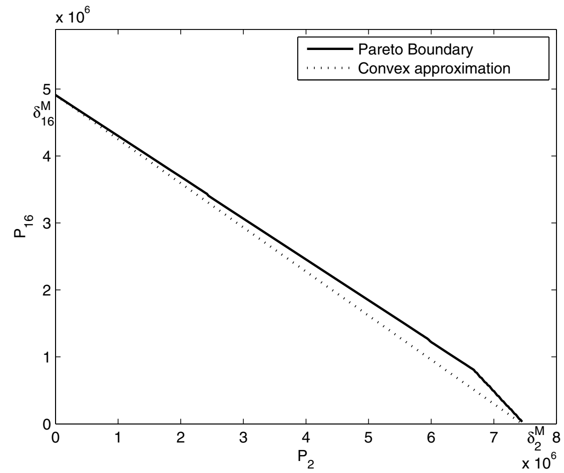

Fig. 2. In this case, only compensators 2 and 16 change their injected power, starting from $P_2 = P_{16} = 0$. The solid line represents the Pareto Front, the dotted line is its approximation by the convex combination of Δ_2 and Δ_{16} (thus shown to be feasible).

to compare the three possible games, we compute the λ_i's playing the games determined by (26), (30), or (31) with $\eta = 0.7$, as well as the values injected, i.e., $\delta_i = \lambda_i \delta_i^M$. The results are reported in Tables II, III, IV, respectively. Table II shows very different values of the δ_i's. In this scenario the compensators closer to the PCC take advantage from their locations. Conversely, the δ_i in Table III are all equal, but there is a pronounced difference among the λ_i's, which implies a dissimilarity in the quantities that each compensators avoid to inject. Furthermore, the total amount of the change in the active power injection, which in some sense represents the "global fairness", is $\sum_{i=1}^{\ell} \delta_i = 2.59$ MW, much lower than the one computed with the δ_i's of Table II (3.30 MW). The δ_i's in Table IV represent globally a mediation between the requirements of "individual fairness" and "global fairness", and sum to 2.88 MW.

As a final consideration, in all the cases the $|u_i'|$'s are all feasible as Proposition 4 holds. In Fig. 2 it is shown how the linear combination of the δ_i's approximate the Pareto front, in the case in which only two compensators play.

VII. CONCLUSIONS

In this paper we analyzed the problem of sharing the active power generation among the micro-generators in a smart grid. We aim at a fair, ethical sharing. First of al,l we characterized a feasible region for the active power injected

TABLE III

RESULTING ALLOCATION BY USING (30)

node	λ	δ [kW]	$\lvert u' \rvert$ [V]
2	0.0433	353.87	4827
4	0.1015	323.76	4845
14	0.1526	323.76	4864
16	0.0659	323.76	4848
20	0.1244	323.76	4866
24	0.1124	323.76	4884
30	0.2005	323.76	4887
33	0.1994	323.76	4900

TABLE IV

RESULTING ALLOCATION BY USING (31) WITH $\eta = 0.7$

node	λ	δ [kW]	$\lvert u' \rvert$ [V]
2	0.0739	552.76	4857
4	0.1360	433.97	4881
14	0.1289	273.37	4854
16	0.0873	428.53	4878
20	0.1282	333.85	4896
24	0.1141	328.81	4893
30	0.1548	249.88	4917
33	0.1767	287.02	4899

by the micro-generators in a smart grid. Then we studied a repeated game setup to reach an agreement among the compensators about the amount of active power that they will inject, moving along a Pareto boundary approximation. We derived and analyzed three types of solutions. Finally we evaluated the resulting approach through simulation over the standard test feeder IEEE37. Future developments are the addition of a voltage magnitude lower bound, a deeper analysis on the creation of active and reactive power feasible injection region, the study of repeated execution of the algorithm in order to better approach the Pareto boundary, the introduction of limitations for the generation capacity of the micro-generators, as well as the algorithm implementation into a real test-bed.

APPENDIX
PROOF OF LEMMA 2

The proof is by induction on n. If $n = 1$, then $\Gamma = a_1 - a_1^2 = a_1(1 - a_1)$. Assume now that the condition is verified up to n, and prove it for $n+1$. We have

$$\Gamma = \begin{bmatrix} a_1 - a_1^2 & \dots & -a_1 a_n & -a_1 a_{n+1} \\ \dots & \dots & \dots & \dots \\ -a_n a_1 & \dots & a_n - a_n^2 & -a_n a_{n+1} \\ -a_{n+1} a_1 & \dots & -a_{n+1} a_n & a_{n+1} - a_{n+1}^2 \end{bmatrix} = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{12}^T & \Gamma_{22} \end{bmatrix}$$

Due to relationship

$$\det \Gamma = \det \Gamma_{11} \det(\Gamma_{22} - \Gamma_{12}^T \Gamma_{11}^{-1} \Gamma_{12})$$

and as it is easy to verify that

$$\Gamma_{11}^{-1} = c \begin{bmatrix} \frac{1-\sum_{i\neq 1} a_i}{a_1} & 1 & \dots & 1 \\ 1 & \frac{1-\sum_{i\neq 2} a_i}{a_2} & \dots & 1 \\ \dots & \dots & \dots & \dots \\ 1 & 1 & \dots & \frac{1-\sum_{i\neq n} a_i}{a_n} \end{bmatrix}$$

with $c = \frac{1}{1-\sum_{i=1}^n a_i}$, then

$$\begin{aligned} \det(\Gamma_{22} - \Gamma_{12}^T \Gamma_{11}^{-1} \Gamma_{12}) &= \\ &= \det\left(a_{n+1} - a_{n+1}^2 - \frac{a_{n+1}^2 \sum_{i=1}^n a_i}{1 - \sum_{i=1}^n a_i} \right) \\ &= \frac{a_{n+1}\left(1 - \sum_{i=1}^{n+1} a_i\right)}{1 - \sum_{i=1}^n a_i} \\ \det \Gamma = \prod_{i=1}^n a_i \Big(1 - \sum_{i=1}^n a_i\Big) & \frac{a_{n+1}\left(1 - \sum_{i=1}^{n+1} a_i\right)}{1 - \sum_{i=1}^n a_i} \\ &= \prod_{i=1}^{n+1} a_i \Big(1 - \sum_{i=1}^{n+1} a_i\Big) \end{aligned}$$

which proves the lemma. ■

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