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Technical Report

Lyapunov Theory for Discrete Time Systems

This work contains a collection of Lyapunov related theorems for discrete time systems. Its main purpose it to collect in a self contained document part of the Lyapunov theory in discrete time, since, in the literature, there does not seem to be a unique work which contains these results and their proof, apart from [2], which deals with discrete time Lyapunov theory, but is written in German, and so it is not easily accessible. The work has been obtained starting from the Lyapunov results for continuous time given in [1] and from the results contained in [2].

Definition 0.1 *A function $f(t,x)$ is said to be* ***Lipschitz*** *in $(\bar{t},\bar{x})$ if*

$$\|f(t,x)-f(t,y)\| \leq L\|x-y\| \tag{1}$$

$\forall\ (t,x),(t,y)$ in a neighbourhood of $(\bar{t},\bar{x})$. The constant L is called ***Lipschitz constant****.*
Consider $f(t,x)=f(x)$ independent of t

- *f is* ***locally Lipschitz*** *on a domain $D\subset\mathbb{R}^n$ open and connected if each point in D has a neighbourhood D_0 such that (1) is satisfied with Lipschitz constant L_0.*
- *f is* ***Lipschitz*** *on a set W if (1) is satisfied for all points in W with the same constant L. A function f locally Lipschitz on D is Lipschitz on every compact subset of D.*
- *f is* ***globally Lipschitz*** *if it is Lipschitz on $\mathbb{R}^n$.*

If $f(t,x)$ depends on t, the same definitions hold, provided that (1) holds uniformly in t for all t in an interval of time (that is the Lipschitz constant do not vary due to the time).

A continuously differentiable function on a domain D is also Lipschitz in the same domain.

1 Autonomous systems

Consider the autonomous system

$$x(t+1)=f(x(t)) \tag{2}$$

where $f\colon D\to\mathbb{R}^n$ is **locally Lipschitz** in $D\subset\mathbb{R}^n$, and suppose $f(0)=0$, that is $x=0$ is an equilibrium point for system (2) (all this can be extended for an equilibrium point different from 0).

Definition 1.1 *The equilibrium point $x = 0$ of (2) is*

- ***stable*** *if, for each $\epsilon > 0$, there is $\delta = \delta(\epsilon)$ such that*

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \ \forall t \geq 0$$

- ***unstable*** *if it is not stable*

- ***asymptotically stable*** *if it is stable and δ can be chosen such that*

$$\|x(0)\| < \delta \Rightarrow \lim_{t \to \infty} x(t) = 0$$

Theorem 1.2 (Existence of a Lyapunov function implies stability) *Let $x = 0$ be an equilibrium point for the autonomous system*

$$x(t+1) = f(x(t))$$

where $f\colon D \to \mathbb{R}^n$ is locally Lipschitz in $D \subset \mathbb{R}^n$ and $0 \in D$. Suppose there exists a function $V\colon D \to \mathbb{R}$ which is continuous and such that

$$V(0) = 0 \text{ and } V(x) > 0, \ \forall x \in D - \{0\} \tag{3}$$
$$V(f(x)) - V(x) \leq 0, \ \forall x \in D \tag{4}$$

Then $x = 0$ is stable. Moreover if

$$V(f(x)) - V(x) < 0, \ \forall x \in D - \{0\} \tag{5}$$

then $x = 0$ is asymptotically stable.

Proof: *Given $\epsilon > 0$, choose $r \in (0, \epsilon]$ such that $B_r = \{x \in \mathbb{R}^n \mid \|x\| < r\} \subset D$. Let $\alpha = \min_{\|x\|=r} V(x)$, then $\alpha > 0$ by (3). Take $\beta \in (0, \alpha)$ and let $\Omega_\beta = \{x \in B_r \mid V(x) \leq \beta\}$. Ω_β is in the interior of B_r, and any trajectory that starts in Ω_β stays in Ω_β for all $t \leq 0$. This is true due to (4), since $V(x(t+1)) \leq V(x(t)) \leq \cdots \leq V(x(0)) \leq \beta, \ \forall t \geq 0$. Since Ω_β is compact and invariant for f, and f is locally Lipschitz in D, there exists a unique solution defined for all $t > 0$ if $x(0) \in \Omega_\beta$. Since $V(x)$ is continuous and $V(0) = 0$, there exists a $\delta > 0$ such that if $\|x\| < \delta$ then $V(x) < \beta$. This implies that $B_\delta \subset \Omega_\beta \subset B_r$, so*

$$x(0) \in B_\delta \Rightarrow x(0) \in \Omega_\beta \Rightarrow x(t) \in \Omega_\beta, \ \forall t \geq 0 \Rightarrow x(t) \in B_r, \ \forall t \geq 0$$

Therefore

$$\|x\| < \delta \Rightarrow \|x(t)\| < r \leq \epsilon, \ \forall t \geq 0$$

and so $x = 0$ is a stable point.
Suppose now that (5) holds. Since $x = 0$ is stable, it is possible to find for every $r > 0$ such that $B_r \subset D$ a constant b such that $\Omega_b \subset B_r$ and all the trajectorys starting from Σ_b stay in Σ_b. Since $V(x)$ is bounded below by 0 and is strictly decreasing along the trajectories in D, it holds that $V(x(t)) \to c \geq 0$ as $t \to \infty$. Suppose ab absurdo that $c > 0$. Starting from a point in Ω_b we have that the trajectories are such that $V(x(t)) \to c$ as $t \to \infty$ with $c < b$ (otherwise if $c = b$ starting from a point $x \in \Omega_b$ such that $V(x) = b$ the trajectory would reach in a step a point such that $V(f(x)) < b$, which is a contradiction). Now, as

before, since $V(x)$ is continuous and $V(0) = 0$, there exists $d > 0$ such that $B_d \subset \Omega_c$. Consider set $\Delta = \{x | d \leq \|x\| \leq r\}$, which is a compact set that contains $\{x | V(x) = c\}$. Since V and f are continuous functions, we can define

$$\gamma := \min_{x \in \Delta} V(x) - V(f(x))$$

due to Bolzano Weierstrass theorem. Since $\lim_{t\to\infty} V(x(t)) = c$ and V is continuous, there exists $\bar{t}$ such that for all $t > \bar{t}$, $V(x(t)) \leq c + \gamma'$, with $\gamma' < \gamma$ and $x(t) \in \Delta$. Since $x(t)$ belongs to Δ, it also holds that $V(x(t)) - V(f(x(t))) \geq \gamma$, and so $V(f(x(t))) \leq -\gamma + V(x(t)) \leq c + \gamma' - \gamma < c$, which is a contradiction. □

Definition 1.3 *A function $V \colon D \to \mathbb{R}$ satisfying* (3) *and* (4) *is called a **Lyapunov function**.*

Theorem 1.4 (Global asymptotic stability from Lyapunov) *Let $x = 0$ be an equilibrium point for the autonomous system*

$$x(t+1) = f(x(t))$$

where $f \colon D \to \mathbb{R}^n$ is locally Lipschitz in $D \subset \mathbb{R}^n$ and $0 \in D$. Let $V \colon \mathbb{R}^n \to \mathbb{R}$ be a continuous function such that

$$V(0) = 0 \text{ and } V(x) > 0, \ \forall x \in D - \{0\} \tag{6}$$
$$\|x\| \to \infty \Rightarrow V(x) \to \infty \tag{7}$$
$$V(f(x)) - V(x) < 0, \ \forall x \in D \tag{8}$$

then $x = 0$ is globally asymptotically stable.

Proof: *Given any point $p \in \mathbb{R}^n$, let $c = V(p)$. Due to (7), for any $c > 0$ there is $r > 0$ such that $V(x) > c$ whenever $\|x\| > r$. Thus $\Omega_c \subset B_r$, and we can proceed as done in Theorem 1.2* □

If an equilibrium point is globally asymptotically stable, than it is the only possible equilibrium point of the system (2).
The following gives a condition for instability.

Theorem 1.5 (Instability condition from Lyapunov) *Let $x = 0$ be an equilibrium point for the autonomous system*

$$x(t+1) = f(x(t))$$

where $f \colon D \to \mathbb{R}^n$ is locally Lipschitz in $D \subset \mathbb{R}^n$ and $0 \in D$. Let $V \colon D \to \mathbb{R}$ be a continuous function such that $V(0) = 0$ and $V(x_0) > 0$ for some x_0 with arbitrary small $\|x_0\|$. Let $r > 0$ be such that $B_r \subset D$ and $U = \{x \in B_r | V(x) > 0\}$, and suppose that $V(f(x)) - V(x) > 0$ for all $x \in U$. Then $x = 0$ is unstable.

Proof: *Consider $x_0 \in U$ and let $a = V(x_0) > 0$. The set $\Sigma_a = \{x \in U | V(x) \geq a\}$ is compact, so we can define $\alpha = \min_{x\in\Sigma}(V(f(x)) - V(x))$. There exists an instant $\bar{t}$ such that $x(t) \in U$ for $0 \leq t < \bar{t}$ and $\|x(\bar{t})\| > r$ for $t = \bar{t}$. This holds because $V(f(x)) > V(x) + \alpha$, $\forall x \in U$ so, at time instant t, $V(x(t+1) =$*

$V(f(x(t))) > V(x(t)) > 0$. Now if $\|x(t+1)\| \leq r$, then the trajectory is still in U, otherwise is in $D \setminus B_r$. The latter is true because to be in $B_r \setminus U$, $V(x(t+1))$ has to be smaller than 0, but $V(x(t+1)) > 0$. So starting from a point arbitrarily close to the origin the trajectory goes outside B_r, and therefore the origin is unstable. □

2 The invariance principle

Definition 2.1 *A point p is said to be a **positive limit point** of $x(t)$ if there is a sequence $\{t_n\}$, with $t_n \to \infty$ as $n \to \infty$ such that $x(t_n) \to p$ as $n \to \infty$. The set of all positive limit points of $x(t)$ is called **positive limit set** of $x(t)$.*

Definition 2.2 *A set M is an **invariant set** with respect to (2) if $x(0) \in M \Rightarrow x(t) \in M, \ \forall\, t \in \mathbb{R}$. It is a **positive invariant set** if $x(0) \in M \Rightarrow x(t) \in M, \ \forall\, t \geq 0$.*

The trajectory $x(t)$ approaches M as $t \to \infty$, if for each $\epsilon > 0$ there is $T > 0$ such that $\text{dist}(x(t), M) < \epsilon, \ \forall\, t > T$, where $\text{dist}(p, M) = \inf_{x \in M} \|p - x\|$. Note that this does not imply that $\lim_{t\to\infty}$ exists.
Equilibrium points and limit cycles are example of invariant sets for (2), and if a Lyapunov function V for the latter system exists, then also the set $\Omega_c = \{x \in D | V(x) \leq c\}$ is an invariant set.

Lemma 2.3 *If a solution $x(t)$ of (2) is bounded and belongs to D for $t \geq 0$, then its positive limit set L^+ is a nonempty, compact, invariant set. Moreover, $x(t)$ approaches L^+ as $t \to \infty$.*

Proof: *The proof can be obtained from appendix C3 of [1]* □

The following is known as LaSalle's theorem

Theorem 2.4 (LaSalle's theorem) *Let $\Omega \subset D$ be a compact set that is positively invariant with respect to the autonomous system*

$$x(t+1) = f(x(t))$$

where $f \colon D \to \mathbb{R}^n$ is locally Lipschitz in $D \subset \mathbb{R}^n$ and $0 \in D$. Let $V : D \to \mathbb{R}$ be a continuous function such that $V(f(x)) - V(x) \leq 0$ in Ω. Let E be the set of all points in Ω where $V(f(x)) - V(x) = 0$, and let M be the largest invariant set in E. Then every solution starting in Ω approaches M as $t \to \infty$.

Proof: *Can be obtained from proof of Theorem 4.4 page 128 in [1].* □

Corollary 2.5 *Let $x = 0$ be an equilibrium point for the autonomous system*

$$x(t+1) = f(x(t))$$

where $f \colon D \to \mathbb{R}^n$ is locally Lipschitz in $D \subset \mathbb{R}^n$ and $0 \in D$. Let $V \colon D \to \mathbb{R}$ be a continuous positive definite function on a domain D, $x \in D$, such that

$V(f(x)) - V(x) \le 0$ in D. Let $S = \{x \in D | V(f(x)) - V(x) = 0\}$ and suppose that no solution can stay identically in S other than the trivial solution $x(t) \equiv 0$. Then the origin is asymptotically stable.

Corollary 2.6 *Let $x = 0$ be an equilibrium point for the autonomous system*

$$x(t+1) = f(x(t))$$

where $f : D \to \mathbb{R}^n$ is locally Lipschitz in $D \subset \mathbb{R}^n$ and $0 \in D$. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuous, positive definite, radially unbounded function, such that $V(f(x)) - V(x) \le 0,\ \forall x \in \mathbb{R}^n$. Let $S = \{x \in \mathbb{R}^n | V(f(x)) - V(x) = 0\}$ and suppose that no solution can stay identically in S other than the trivial solution $x(t) \equiv 0$. Then the origin is **globally** *asymptotically stable.*

LaSalle principle is useful because

- it gives an estimate of the region of attraction of the equilibrium point. It can be any compact positively invariant set;
- there is an equilibrium set and not an isolated equilibrium point;
- function $V(x)$ does not have to be positive definite;
- in case of the corollaries it relaxes the negative definiteness on $V(f(x)) - V(x)$.

Before going to the linearisation part, we prove the following theorem concerning systems with exponentially asymptotic equilibrium points (see [1, Ex. 4.68])

Theorem 2.7 (Exponential stability implies existence of a Lyapunov function) *Let $x = 0$ be an equlibrium point for the nonlinear system the autonomous system*

$$x(t+1) = f(x(t))$$

where $f : D \to \mathbb{R}^n$ is continuously differentiable and $D = \{x \in \mathbb{R}^n | \ \|x\| < r\}$. Let k, λ, and r_0 be positive constants with $r_0 < r/k$. Let $D_0 = \{x \in \mathbb{R}^n | \ \|x\| < r_0\}$. Assume that the solutions of the system satisfy

$$\|x(t)\| \le k\|x(0)\|e^{-\lambda t},\ \forall x(0) \in D_0,\ \forall t \ge 0 \tag{9}$$

Show that there is a function $V : D_0 \to \mathbb{R}$ that satisfies

$$\begin{gathered} c_1\|x\|^2 \le V(x) \le c_2\|x\|^2 \\ V(f(x)) - V(x) \le -c_3\|x\|^2 \\ |V(x) - V(y)| \le c_4\|x - y\|(\|x\| + \|y\|) \end{gathered}$$

for all $x, y \in D_0$ and for some positive constants c_1, c_2, c_3 and c_4.

Proof: *Let $\phi(t, x)$ be the solution of $x(t+1) = f(x(k))$ at time t starting from $x(0) = x$ at time $k = 0$. Let*

$$V(x) = \sum_{t=0}^{N-1} \phi^\top(t, x)\phi(t, x)$$

for some integer variable N to be set. Then

$$V(x) = x^\top x + \sum_{t=1}^{N-1} \phi^\top(t,x)\phi(t,x) \geq x^\top x = \|x\|$$

and on the other hand, using (9) we have

$$V(x) = \sum_{t=0}^{N-1} x(t)^\top x(t) \leq \sum_{t=0}^{N-1} k^2\|x\|^2 e^{-2\lambda t} \leq k^2 \left(\frac{1-e^{-2\lambda N}}{1-e^{-2\lambda}}\right)\|x\|^2$$

We have shown that there exists c_1 and c_2 such that

$$c_1\|x\|^2 \leq V(x) \leq c_2\|x\|^2$$

is satisfied. Now, since $\phi(t, f(x)) = \phi(t, \phi(1,x)) = \phi(t+1,x)$,

$$\begin{aligned}
V(f(x)) - V(x) &= \sum_{t=0}^{N-1} \phi^\top(t+1,x)\phi(t+1,x) - \sum_{t=0}^{N-1} \phi^\top(t,x)\phi(t,x) = \\
&= \sum_{j=1}^{N} \phi^\top(j,x)\phi(j,x) - \sum_{t=0}^{N-1} \phi^\top(t,x)\phi(t,x) = \phi^\top(N,x)\phi(N,x) - x^\top x \\
&\leq k^2 e^{-2\lambda N}\|x\|^2 - \|x\|^2 = -(1-k^2e^{-2\lambda N})\|x\|^2
\end{aligned}$$

Now we can choose N big enough so that $1 - k^2e^{-2\lambda N}$ is greater than 0 and also the second property has been proven. For the third property, since f is continuously differentiable it is also Lipschitz over the bounded domain D, with a Lipschitz constant L, for which it holds $\|f(x) - f(y)\| \leq L\|x-y\|$. Then

$$\|\phi(t+1,x) - \phi(t+1,y)\| = \|f(\phi(t,x)) - f(\phi(t,y))\| \leq L\|\phi(t,x) - \phi(t,y)\|$$

and by induction

$$\|\phi(t,x) - \phi(t,y)\| \leq L^t\|x-y\|$$

Consider now

$$\begin{aligned}
|V(x) - V(y)| &= \left|\sum_{t=0}^{N-1} (\phi^\top(t,x)\phi(t,x) - \phi^\top(t,y)\phi(t,y))\right| \\
&= \left|\sum_{t=0}^{N-1} [\phi^\top(t,x)(\phi(t,x) - \phi(t,y)) + \phi^\top(t,y)(\phi(t,x) - \phi(t,y))]\right| \\
&\leq \sum_{t=0}^{N-1} [\|\phi^\top(t,x)\|\|\phi(t,x) - \phi(t,y)\| + \|\phi^\top(t,y)\|\|\phi(t,x) - \phi(t,y)\|] \\
&\leq \sum_{t=0}^{N-1} [\|\phi^\top(t,x)\| + \|\phi^\top(t,y)\|]L^t\|x-y\| \\
&\leq \left[\sum_{t=0}^{N-1} ke^{-\lambda t}L^t\right](\|x\| + \|y\|)\|x-y\| \\
&\leq c_4(\|x\| + \|y\|)\|x-y\|
\end{aligned}$$

and so we have proven the last inequality. □

3 Linear systems and Linearization

Consider the linear time-invariant system

$$x(t+1) = Ax(t), \;\; A \in \mathbb{R}^{n\times n} \tag{10}$$

It has an equilibrium point in the origin $x = 0$. The solution of the linear system starting from $x_0 \in \mathbb{R}^n$ has the form

$$x(t) = A^t x(0)$$

We have the following result on the stability of linear systems

Theorem 3.1 *The equilibrium point $x = 0$ of the linear time-invariant system*

$$x(t+1) = Ax(t), \;\; A \in \mathbb{R}^{n\times n}$$

is stable if and only if all the eigenvalues of A satisfy $|\lambda_i| \leq 1$ and the algebraic and geometric multiplicity of the eigenvalues with absolute value 1 coincide. The equilibrium point $x = 0$ is globally asymptotically stable if and only if all the eigenvalues of A are such that $|\lambda_i| < 1$.

A matrix A with all the eigenvalues in absolute value smaller than 1 is called a **Schur matrix**, and it holds that the origin is asymptotically stable if and only if matrix A is Schur.

To use Lyapunov theory for linear system we can introduce the following candidate

$$V(x) = x^\top P x$$

with P a symmetric positive definite matrix. It's total difference is

$$V(f(x)) - V(x) = x^\top A^\top P A x - x^\top P x = x^\top (A^\top P A - P)x := -x^\top Q x$$

Using Theorem 1.2 we have that if Q is positive-semidefinite the origin is stable, whether if Q is positive definite the origin is asymptotically stable. Fixing a positive definite matrix Q, if the solution of the Lyapunov equation

$$A^\top P A - P = -Q \tag{11}$$

with respect to P is positive definite, then the trajectories converge to the origin.

Theorem 3.2 (Lyapunov for linear time invariant systems) *A matrix A is Schur if and only if, for any positive definite matrix Q there exists a positive definite symmetric matrix P that satisfies* (11)*. Moreover if A is Schur, then P is the unique solution of* (11)*.*

Proof: *Sufficiency can be obtained combining Theorem 3.1 and the fact that the existence of solution P for any positive definite matrix Q assures the convergence of the trajectory. Suppose now that A is Schur stable and build matrix P as*

$$P = \sum_{t=0}^{\infty} (A^\top)^t Q A^t \tag{12}$$

for any positive definite Q. Matrix P is symmetric and positive definite since Q is positive definite. We need to show that using this P equation (11) is satisfied. Substituting (12) in (11) we obtain

$$A^\top \sum_{t=0}^{\infty}\left[(A^\top)^t Q A^t\right] A - \sum_{t=0}^{\infty}(A^\top)^t Q A^t = \sum_{t=0}^{\infty}\left[(A^\top)^{t+1} Q A^{t+1} - (A^\top)^t Q A^t\right] = -Q$$

Suppose now that P is not unique, so there exists a $\tilde{P} \neq P$ such that (11) is satisfied. So it holds

$$A^\top P A - P = A^\top \tilde{P} A - \tilde{P} \Leftrightarrow A^\top (P - \tilde{P}) A - (P - \tilde{P}) = 0$$

from which, defining $R(x) := x^\top (P - \tilde{P}) x$ it follows that

$$R(f(x)) - R(x) = 0,\ \forall x \in \mathbb{R}^n \Leftrightarrow R(x(0)) = R(x(t)),\ \forall t \geq 0.$$

Now it holds that

$$\lim_{t\to\infty} R(x(t)) = \lim_{t\to\infty} x(0)^\top (A^\top)^t (P - \tilde{P}) A^t x(0) = 0,\ \forall x \in \mathbb{R}^n$$

since A is Schur stable, so due to the fact that $R(x(0)) = R(x(t))$, we have that

$$x^\top (P - \tilde{P}) x = 0,\ \forall x \in \mathbb{R}^n \Leftrightarrow P - \tilde{P} = 0$$

and so the solution is unique. □

Let us consider again the nonlinear model

$$x(t+1) = f(x(t))$$

with $f\colon D \to \mathbb{R}^n$ a continuously differentiable map from $D \subset \mathbb{R}^n,\ 0 \in D$ into $\mathbb{R}^n$ such that $f(0) = 0$. Using the mean value theorem, each component of f can be rewritten in the following form

$$f_i(x) = \frac{\partial f_i}{\partial x}(z_i) x$$

for some z_i on the segment from the origin to x. It is valid for any $x \in D$, where the line connecting x to the origin entirely belongs to D. We can also write

$$f_i(x) = \frac{\partial f_i}{\partial x}(0) x + \underbrace{\left[\frac{\partial f_i}{\partial x}(z_i) - \frac{\partial f_i}{\partial x}(0)\right] x}_{g_i(x)}$$

where each $g_i(x)$ satisfies

$$|g_i(x)| \leq \left\| \frac{\partial f_i}{\partial x}(z_i) - \frac{\partial f_i}{\partial x}(0) \right\| \|x\|$$

Function f can be rewritten as

$$f(x) = Ax + g(x)$$

where $A = \frac{\partial f_i}{\partial x}(0)$. By continuity of $\frac{\partial f_i}{\partial x}$ we have that

$$\frac{\|g(x)\|}{\|x\|} \to 0 \text{ as } \|x\| \to 0$$

Therefore, in a small neighbourhood of the origin the nonlinear system can be approximated by $x(t+1) = Ax(t)$.

The following is known as Lyapunov's indirect method.

Theorem 3.3 (Linearised asympt stable implies nonlin asympt stable) *Let $x = 0$ be an equilibrium point for the nonlinear autonomous system*

$$x(t+1) = f(x(t))$$

where $f\colon D \to \mathbb{R}^n$ is locally Lipschitz in $D \subset \mathbb{R}^n$ and $0 \in D$. Let $A = \frac{\partial f_i}{\partial x}(x)\Big|_{x=0}$. Then the origin is asymptotically stable if $|\lambda_i| < 0$ for all the eigenvalues of A. Instead, if there exists at least an eigenvalue such that $|\lambda_i| > 0$, then the origin is unstable.

Proof: *Since A is stable there exists a positive definite matrix P, $p_1 I \le P \le p_2 I$, such that $V(x)$ is a Lyapunov function for the linearised system $x(t+1) = Ax(t)$, and so it solves (11) for any positive definite matrix Q, $q_1 I \le Q \le q_2 I$. Applying the same Lyapunov function to the nonlinear system we get the following total difference*

$$\begin{aligned} V(f(x)) - V(x) &= f(x)^\top P f(x) - x^\top P x = (Ax + g(x))^\top P(Ax + g(x)) - x^\top P x \\ &= x^\top A^\top P A x - x^\top P x + 2g(x)^\top P x + g(x)^\top P g(x) \\ &= -x^\top Q x + 2g(x)^\top P x + g(x)^\top P g(x) \end{aligned}$$

Since $\frac{\|g(x)\|}{\|x\|} \to 0$ as $\|x\| \to 0$, fixed a constant $\gamma > 0$, there exists a neighbourhood of x, $\|x\| < \epsilon$ such that $\|g(x)\| < \gamma\|x\|$, and so

$$V(f(x)) - V(x) \le -q_1\|x\|^2 + p_2\gamma^2\|x\|^2 + 2p_2\gamma\|x\|^2 = (-q_1 + p_2\gamma^2 + 2p_2\gamma)\|x\|^2, \ \forall \|x\| < \epsilon$$

Therefore, choosing γ such that $-q_1 + p_2\gamma^2 + 2p_2\gamma$ is negative, $V(x)$ is indeed a Lyapunov function for the starting nonlinear system. Note that $-q_1 + p_2\gamma^2 + 2p_2\gamma = 0$ describes in γ a parabola whose vertex is in the third quarter and is directed towards the upper part of the plane, so there exists a γ which satisfy the property required.

To show the instability part, we first give the following statement regarding the solvability of the discrete Lyapunov equation

Lemma 3.4 *B The Lyapunov equation (11) admits a solution if and only if the eigenvalues λ_i of matrix A are such that*

$$\lambda_i \lambda_j \ne 1 \text{ for all } i, j = 1, \ldots, n \tag{13}$$

Moreover given a positive definite matrix Q, the corresponding solution P is positive definite if and only if $|\lambda_i| < 1$ for all $i = 1, \ldots, n$.

The proof of the previous lemma can be found in [2]. Suppose now that there is λ_i such that $|\lambda_i| > 1$ but that condition (13) is satisfied. Therefore, given a positive definite matrix Q, the corresponding solution P of $A^\top PA - P = -Q$ is not positive semi-definite (note that if Q is positive definite and A is Schur stable, then P cannot be positive semi-definite, since if x is such that $x^\top Px = 0$, then $x^\top A^\top PAx < 0$ but this is not possible since P is positive semi-definite). Matrix $\tilde{P} = -P$ is not negative semi-definite, so defining $V(x) = x^\top \tilde{P}x$, it holds $V(x) > 0$ for some $x \in \mathbb{R}^n$; for the same vector x it also holds

$$\begin{aligned} V(f(x)) - V(x) = x^\top A^\top \tilde{P}Ax - x^\top \tilde{P}x = x^\top (A^\top \tilde{P}A - \tilde{P})x = \\ - x^\top (A^\top PA - P)x = -x^\top(-Q)x = x^\top Qx > 0 \end{aligned}$$

Now we can apply Theorem 1.5 and conclude that the equilibrium point is instable. If condition (13) is not satisfied, consider matrix $A_1 = \frac{1}{\gamma}A$, with γ such that (13) is satisfied using matrix A_1 and the solution matrix P_1 for (11) for any positive definite matrix Q is not positive semi-definite (that is there is at least one eigenvalue of A_1 with modulus greater than 1). Therefore, choosing $V(x) = -x^\top P_1 x$, it holds that $V(x) > 0$ for some x. It also holds that

$$\begin{aligned} V(f(x)) - V(x) &= -x^\top A^\top P_1 Ax + x^\top P_1 x = -x^\top(\gamma^2 A_1^\top P_1 A_1 - P_1 - \gamma^2 P_1 + \gamma^2 P_1)x \\ &= -\gamma^2 x^\top (A_1^\top P_1 A_1 - P_1)x - (\gamma^2 - 1)x^\top P_1 x \\ &= -\gamma^2 x^\top(-Q)x - (\gamma^2 - 1)x^\top P_1 x = \gamma^2 x^\top Qx - (\gamma^2 - 1)x^\top P_1 x \end{aligned}$$

Choosing an adequately small γ it holds $V(f(X)) - V(x) > 0$ and we can apply again Theorem 1.5. □

This theorem allows to find a Lyapunov function for the nonlinear system in a neighbourhood of the origin, provided that the linearised system is asymptotically stable.

4 Comparison Functions

Consider the nonautonomous system

$$x(t+1) = f(t, x(t))$$

starting from $x(t_0) = x_0$ at time t_0, with $f\colon (\mathbb{T} \times D) \to \mathbb{R}^n$, $\mathbb{T} = \{t_0, t_0 + 1, \dots\}$, $D \subset \mathbb{R}^n$. The evolution of the system depends on the starting time t_0. We need new definitions for the stability in order to have them hold uniformly in the initial time t_0. We will exploit the following class of functions

Definition 4.1 *A continuous function $\alpha\colon [0, a) \to [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$.*

Definition 4.2 *A continuous function $\beta\colon [0, a) \times [0, \infty) \to [0, \infty)$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r and, for each fixed r, the mapping $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to 0$.*

Lemma 4.3 *Let α_1 and α_2 be class $\mathcal{K}$ functions on $[0, a)$, α_3 and α_4 be class $\mathcal{K}_\infty$ functions, and β be a class $\mathcal{KL}$ function. Denote the inverse of α_i by α_i^{-1}. Then*

- α_1^{-1} *is defined on $[0, \alpha_1(a))$ and belongs to class $\mathcal{K}$.*
- α_3^{-1} *is defined on $[0, \infty)$ and belongs to class $\mathcal{K}_\infty$.*
- $\alpha_1 \circ \alpha_2$ *belongs to class $\mathcal{K}$*
- $\alpha_3 \circ \alpha_4$ *belongs to class $\mathcal{K}_\infty$*
- $\sigma(r, s) = \alpha_1(\beta(\alpha_2(r), s))$ *belongs to class $\mathcal{KL}$.*

These classes of functions are connected to the Lyapunov theory for autonomous systems through these Lemmas

Lemma 4.4 *Let $V\colon D \to \mathbb{R}$ be a continuous positive definite function defined on a domain $D \subset \mathbb{R}^n$ that contains the origin. Let $B_r \subset D$ for some $r > 0$. Then there exist class $\mathcal{K}$ functions α_1 and α_2, defined on $[0, r)$, such that*

$$\alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|)$$

for all $x \in B_r$. If $D = \mathbb{R}^n$, the functions α_1 and α_2 will be defined on $[0, \infty)$ and the previous inequality will hold for all $x \in \mathbb{R}^n$. Moreover, if $V(x)$ is radially unbounded, then α_1 and α_2 can be chosen to belong to class $\mathcal{K}_\infty$.

Proof: Corollary C.4 in [1]. □

If V is a quadratic positive definite function $V(x) = x^\top P x,\ P > 0$, then the previous lemma follows from the fact that

$$\lambda_{\min}(P) I \le V(x) \le \lambda_{\max}(P) I$$

5 Nonautonomous Systems

Consider the nonautonomous system

$$x(t+1) = f(t, x(t)) \tag{14}$$

starting from $x(t_0) = x_0$ at time t_0, with $f\colon (\mathbb{T} \times D) \to \mathbb{R}^n$, $\mathbb{T} = \{t_0, t_0 + 1, \dots\}$, $0 \in D \subset \mathbb{R}^n$ locally Lipschitz in x on $T \times D$. The origin is an equilibrium point for (14) if

$$f(t, 0) = 0,\ \forall t \in \mathbb{T}$$

Definition 5.1 *The equilibrium point $x = 0$ of (14) is*

- ***stable*** *if, for each $\epsilon > 0$, there is $\delta = \delta(\epsilon, t_0) > 0$ such that*

$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \epsilon,\ \forall t \ge t_0 \ge 0 \tag{15}$$

- ***uniformly stable*** *if, for each $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$, independent of t_0, such that (15) is satisfied*
- ***unstable*** *if it is not stable*

- ***asymptotically stable*** *if it is stable and there is a positive constant* $c = c(t_0)$ *such that* $x(t) \to 0$ *as* $t \to \infty$*, for all* $\|x(t_0)\| < c$.
- ***uniformly asymptotically stable*** *if it is uniformly stable and there is a positive constant c, independent of* t_0*, such that* $x(t) \to 0$ *as* $t \to \infty$*, for all* $\|x(t_0)\| < c$ *uniformly in* t_0*; that is, for each* $\eta > 0$*, there is* $T = T(\eta) > 0$ *such that*

$$\|x(t)\| < \eta, \ \forall \ t > t_0 + T(\eta), \ \forall \|x(t_0)\| < c$$

- ***globally uniformly asymptotically stable*** *if it is uniformly stable,* $\delta(\epsilon)$ *can be chosen to satisfy* $\lim_{\epsilon \to \infty} \delta(\epsilon) = \infty$*, and, for each pair of positive numbers* η *and c, there is* $T = T(\eta, c) > 0$ *such that*

$$\|x(t)\| < \eta, \ \forall \ t > t_0 + T(\eta, c), \ \forall \|x(t_0)\| < c$$

Lemma 5.2 (Stability definition through class $\mathcal{K}$ functions) *The equilibrium point* $x = 0$ *of* $x(t+1) = f(t,x)$ *is*

- *uniformly stable if and only if there exists a class* $\mathcal{K}$ *function* α *and a positive constant c, independent of* t_0*, such that*

$$\|x(t)\| \leq \alpha(\|x(t_0)\|), \ \forall \ t \geq t_0 \geq 0, \ \forall \|x(t_0)\| < c$$

- *uniformly asymptotically stable if and only if there exist a class* $\mathcal{KL}$ *function* β *and a positive constant c, independent of* t_0*, such that*

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \ \forall \ t \geq t_0 \geq 0, \ \forall \|x(t_0)\| < c \tag{16}$$

- *globally uniformly stable if and only if inequality* (16) *is satisfied for any initial state* $x(t_0)$*.*

Proof: *Proof in Appendix C.6 in [1].* □

An important case for an uniformly asymptotically stable point is when $\beta(r, s) = kre^{-\lambda s}$, with $\lambda > 0$. In this case we have the following

Definition 5.3 *The equilibrium point* $x = 0$ *of* (14) *is called* ***exponentially stable*** *if there exist positive constants* c, k *and* λ *such that it holds*

$$\|x(t)\| \leq k\|x(t_0)\|e^{-\lambda(t-t_0)}, \ \forall \|x(t_0)\| < c \tag{17}$$

and is said to be ***globally exponentially stable*** *if the previous inequality holds for any initial state* $x(t_0)$*.*

Note that since λ is positive, $e^{-\lambda(t-t_0)}$ is equivalent to γ^{t-t_0} with $\gamma = e^{-\lambda} < 1$.

Theorem 5.4 (Lyapunov function implies stability for nonautonomous) *Let* $x = 0$ *be an equilibrium point for the nonautonomous system*

$$x(t+1) = f(t, x(t))$$

with $f\colon (\mathbb{T} \times D) \to \mathbb{R}^n$, $0 \in D \subset \mathbb{R}^n$ *locally Lipschitz in x on* $\mathbb{T} \times D$*. Let* $V\colon \mathbb{T} \times D \to \mathbb{R}$ *be a continuous function such that*

$$W_1(x) \leq V(t,x) \leq W_2(x)$$
$$V(t+1, f(t,x)) - V(t,x) \leq 0$$

for all $t \geq 0$ *and for all* $x \in D$*, where* $W_1(x)$ *and* $W_2(x)$ *are continuous positive definite functions on D. Then* $x = 0$ *is uniformly stable.*

Proof: Choose $r > 0$ and $c > 0$ such that $B_r \subset D$ and $c < \min_{\|x\|=r} W_1(x)$. Then $\{x \in B^r \mid W_1(s) \leq c\}$ is in the interior of B_r. Define $\Omega_{t,c}$ as

$$\Omega_{t,c} = \{x \in B_r | V(t,x) \leq c\}$$

The set $\Omega_{t,c}$ contains $\{x \in B^r \mid W_2(s) \leq c\}$, since $W_2(x) \leq c \Rightarrow V(t,x) \leq c$; for similar reasons $\Omega_{t,c} \subset \{x \in B^r \mid W_1(s) \leq c\}$. So we have

$$\{x \in B^r \mid W_2(s) \leq c\} \subset \Omega_{t,c} \subset \{x \in B^r \mid W_1(s) \leq c\} \subset B_r \subset D$$

for all $t \geq 0$. Since $V(t+1,x) - V(t,x) \leq 0$ in D, for any $t_0 \geq 0$ and $x(t_0) \in \Omega_{t_0,c}$, the solution starting at $(t_0, x(t_0))$ will stay in $\Omega_{t,c}$ for all $t \geq t_0$. We have shown that a solution is bounded and defined for all $t \geq t_0$. We now use Lemma 5.2 (we still don't have sets defined on the norm of vector x). Due to the second property of V we have

$$V(t, x(t)) \leq V(t_0, x(t_0)), \ \forall \ t \geq t_0$$

Since W_1 and W_2 are positive definite matrix, due to lemma 4.4 there are class $\mathcal{K}$ functions α_1 and α_2 defined in $[0, r)$ such that

$$\alpha_1(\|x\|) \leq W_1(x) \leq V(t,x) \leq W_2(x) \leq \alpha_2(\|x\|) \Rightarrow \alpha_1(\|x\|) \leq V(t,x) \leq \alpha_2(\|x\|)$$

So we have (note that α_1 is smaller than V, so to reach the same value of V, the argument of α_1 has to be greater than the norm of the vector in V)

$$\|x(t)\| \leq \alpha_1^{-1}(V(t, x(t))) \leq \alpha_1^{-1}(V(t_0, x(t_0)) \leq \alpha^{-1}(\alpha_2(\|x(t_0)\|))$$

Since $\alpha_1^{-1}(\alpha_2(x))$ is a class $\mathcal{K}$ function we are done. □

Theorem 5.5 (Lyapunov function for asymptotically stable nonautonomous systems) *Suppose the assumptions of Theorem 5.4 are satisfied and that it also holds*

$$V(t+1, f(x,t)) - V(t,x) \leq -W_3(x), \ \forall \ t \geq 0, x \in D$$

where $W_3(x)$ is a continuous positive definite function on D. Then, $x = 0$ is uniformly asymptotically stable.

Proof: Consider $r > 0$ such that $B_r \subset D$. Due to theorem 4.4, there exist class $\mathcal{K}$ functions $\alpha_1, \alpha_2, \alpha_3$ on $[0, r)$ such that

$$\alpha_1(\|x\|) \leq V(t,x) \leq \alpha_2(\|x\|) \quad V(t+1, f(x,t)) - V(t,x) \leq -\alpha_3(\|x\|)$$

For any fixed ϵ, $r \geq \epsilon > 0$, there exists a positive constant $\delta \leq \epsilon$ that satisfy the stability property. Consider a value η, $0 < \eta < \epsilon$ and define $\Lambda = \alpha_1(\eta)$. Consider $x(t_0) \in B_\delta$; if $V(t_0, x(t_0)) < \Lambda$, then the definition is already satisfied since $\|x(t_0)\|$ is necessarily smaller then η and V is contracting along the trajectory. If $x(t_0)$ satisfies $V(t_0, x(t_0)) \geq \Lambda$, define $\Gamma = \alpha_2^{-1}(\Lambda) > 0$ (note that $\Gamma \leq \eta$). It follows that if x is such that $V(t,x) \geq \Lambda$, then $\|x\| \geq \Gamma$. Let $\Omega = \{x \mid \Gamma \leq \|x\| \leq \epsilon\}$, which is closed and bounded. For all $t \geq t_0$ such that $V(t, x(t))$ is greater than Λ, it also holds $\|x(t)\| \geq \Gamma > 0$, so $\alpha_3(\|x(t)\|) \geq \alpha_3(\Gamma)$ which implies

$-\alpha_3(\|x(t)\|) \leq -\alpha_3(\Gamma)$. So we have $V(t+1, f(x,t)) - V(t,x) \leq -\alpha_3(\|x\|) \leq -\alpha(\Gamma) < 0$. So it holds $\forall k \geq 0 \mid \tilde{V}(t_0 + k) \geq \Lambda$

$$\tilde{V}(t_0 + k) = \tilde{V}(t_0) + \sum_{i=0}^{k-1} \left[\tilde{V}(t_0 + i + 1) - \tilde{V}(t_0 + i)\right] \leq \tilde{V}(t_0) - k\alpha_3(\Gamma)$$

This shows that there exists a $\bar{k}$, that depends on η and on δ but not on t_0, such that $\Lambda \leq \tilde{V}(t_0 + \bar{k}) < \Lambda + \alpha_3(\Gamma)$, from which it follows that $\tilde{V}(t_0 + \bar{k} + 1) < \Lambda$ and as already discussed $\|x(t_0 + k)\| < \eta$ for all $k \geq \bar{k} + 1$. □

Definition 5.6 *A function $V(t,x)$ is said to be*

- ***positive semidefinite*** *if $V(t,x) \geq 0$*
- ***positive definite*** *if $V(t,x) \geq W_1(x)$ with W_1 positive definite*
- ***radially unbounded*** *if $W_1(x)$ is so*
- ***decrescent*** *if $V(t,x) \leq W_2(x)$ with W_2 positive definite.*

Theorem 5.7 (Lyap exponentially bounded implies exponential stab) *Let $x = 0$ be an equilibrium point for the nonautonomous system*

$$x(t+1) = f(t, x(t))$$

with $f : (\mathbb{T} \times D) \to \mathbb{R}^n$, $0 \in D \subset \mathbb{R}^n$ locally Lipschitz in x on $\mathbb{T} \times D$. Let $V : \mathbb{T} \times D \to \mathbb{R}$ be a positive definite continuous on x function such that

$$V(t,x) < a\|x\|^2$$
$$\Delta(t,x) := V(t+1, f(t,x)) - V(t,x) \leq -b\|x\|^2$$

for all $t \geq 0$ and for all $x \in D$, where a and b are positive constants. Then $x = 0$ is exponentially stable. If the assumptions hold globally, then the equlibrium point is globally exponentially stable.

Proof: For any given trajectory of the system starting from $x_0 \in D$, due to the assumptions it holds that

$$\Delta(t, x(t)) \leq -(b/a)V(t, x(t)) \leq -cV(t, x(t)),\ 0 < c < 1$$

Exploiting the definition of Δ, we have

$$V(t, x(t)) \leq (1-c)V(t-1, x(t-1)) \leq \cdots \leq (1-c)^{t-t_1} V(t_1, x(t_1)),\ t \geq t_1 \geq t_0$$

Since $c < 1$, then $(1-c)^{t-t_1} = e^{-\gamma(t-t_1)}$, $\gamma > 0$, so

$$V(t, x(t)) \leq e^{-\gamma(t-t_1)} V(t_1, x(t_1))$$

Fixing $t_1 = t_0 + p$, p can be chosen such that $V(t_1, x(t_1)) \leq d\|x(t_0)\|$, with d independent of $x(t_0)$. Now, combining the previous results,

$$\Delta(t, x(t)) \leq -cV(t, x(t)) \leq -ce^{-\gamma(t-t_1)} V(t_1, x(t_1))$$

from which it follows

$$\|x(t)\| \leq -(1/b)\Delta(t, x(t)) \leq \frac{c}{b} e^{-\gamma(t-t_1)} V(t_1, x(t_1)) \leq \frac{dc}{b} e^{-\gamma(t-t_0)} \|x(t_0)\| e^{\gamma p}$$

which has the form of equation (17) □

Theorem 5.8 (Exp stability assures presence Lyap funct nonauton) *Let $x = 0$ be an equilibrium point for the system*

$$x(t+1) = f(t, x)$$

where $f\colon \mathbb{T} \times D \to \mathbb{R}^n$ is locally Lipschitz in x on $\mathbb{T} \times D$, and $D = \{x | \|x\| < r\}$. If there exist positive constants $k, c, c < r/k$ and $\lambda, \lambda < 1$ such that for any intial $x(t_0)$ in $B_c = \{x | \|x\| < c\} \subset D$ the equilibrium point is exponentially stable, that is

$$\|x(t)\| \leq k \|x(t_0)\| e^{-\lambda(t-t_0)}, \;\; \forall \|x(t_0)\| \in B_c$$

then there exists a Lyapunov function $V(t, x)$ for the system. The latter satisfies the following inequalities

$$c_1 \|x\|^2 \leq V(t, x) \leq c_2 \|x\|^2$$
$$V(t+1, f(t, x)) - V(t, x) \leq -c_3 \|x\|^2$$
$$|V(t, x) - V(t, y)| \leq c_4 \|x - y\| (\|x\| + \|y\|)$$

for all $x, y \in B_\delta$ and for some positive constants c_1, c_2, c_3 and c_4.

Proof: *Let $\phi(t, t_0; x)$ be the solution of $x(t+1) = f(t, x(t))$ at time t starting from $x(t_0) = x$ at time t_0. It holds $\phi(t_0, t_0; x) = x$. Let*

$$V(t, x) = \sum_{k=t}^{N-1+t} \phi(k, t; x)^\top \phi(k, t; x)$$

for some integer variable N to be set. Then

$$V(t, x) = x^\top x + \sum_{k=t+1}^{N-1+t} \phi(k, t; x)^\top \phi(k, t; x) \geq x^\top x = \|x\|$$

and on the other hand, due to the exponential stability we have

$$V(t, x) = \sum_{k=t}^{N-1+1} \phi(k, t; x)^\top \phi(k, t; x) \leq \sum_{\tau=t}^{N-1+t} k^2 \|x\|^2 e^{-2\lambda(\tau - t)} \leq k^2 \left(\frac{1 - (e^{-2\lambda})^N}{1 - e^{-2\lambda}} \right) \|x\|^2$$

We have shown that there exists c_1 and c_2 such that

$$c_1 \|x\|^2 \leq V(t, x) \leq c_2 \|x\|^2$$

is satisfied. Now, since $\phi(t+1+k, t+1; f(t,x)) = \phi(t+1+k, t+1; \phi(t+1,t;x)) = \phi(t+1+k,t;x)$,

$$\begin{aligned}
V(t+1, & f(t,x)) - V(t,x) = \\
&= \sum_{k=t+1}^{N-1+t+1} \phi(k,t+1;f(t,x))^\top \phi(k,t+1;f(t,x)) - \sum_{k=t}^{N-1+t} \phi(k,t;x)^\top \phi(k,t;x) = \\
&= \sum_{\Delta=0}^{N-1} \phi(t+1+\Delta,t;x)^\top \phi(t+1+\Delta,t;x) - \sum_{\Delta=0}^{N-1} \phi(t+\Delta,t;x)^\top \phi(t+\Delta,t;x) = \\
&= \phi(t+N,t;x)^\top \phi(t+N,t;x) - \phi(t,t;x)^\top \phi(t,t;x) \\
&\leq k^2 e^{-2\lambda N}\|x\|^2 - \|x\|^2 = -(1-k^2e^{-2\lambda N})\|x\|^2
\end{aligned}$$

Now we can choose N *big enough so that* $1-k^2e^{-2\lambda N}$ *is greater than* 0 *and also the second property has been proven. For the third property, since* B_c *is a compact set, function* $f(t,x)$ *is Lipschitz in* B_δ *uniformly in* t*, with a Lipschitz constant* L*, so it holds* $\|f(t,x)-f(t,y)\| \leq L\|x-y\|\ \forall t \in \mathbb{T}$*. Then*

$$\begin{aligned}
\|\phi(t+\Delta+1,t;x) - \phi(t+\Delta+1,t;y)\| &= \|f(t+\Delta,\phi(t+\Delta,t;x)) - f(t+\Delta,\phi(t+\Delta,t;y))\| \\
&\leq L\|\phi(t+\Delta,t;x) - \phi(t+\Delta,t;y)\|
\end{aligned}$$

and by induction

$$\|\phi(t+\Delta,t;x) - \phi(t+k,t;y)\| \leq L^\Delta \|x-y\|$$

Proceeding as in the proof of Theorem 2.7 we have

$$\begin{aligned}
|V(t,x)-V(t,y)| =&\leq \sum_{k=t}^{N-1+t} [\|\phi^\top(k,t;x)\| + \|\phi^\top(k,t;y)\|] L^k \|x-y\| \\
&\leq \left[\sum_{\tau=t}^{N-1+t} k e^{-\lambda\tau} L^k\right] (\|x\|+\|y\|)\|x-y\| \\
&\leq c_4(\|x\|+\|y\|)\|x-y\|
\end{aligned}$$

and so we have proven the last inequality. □

5.1 Linear systems and Linearisation

Consider now the linear time variant system

$$x(t+1) = A(t)x(t) \tag{18}$$

which has an equilibrium point in the origin. The following theorem holds

Theorem 5.9 (Lyapunov function for linear time variant syst) *Consider the system* (18)*. If there exists a continuous, symmetric, bounded positive definite matrix* $P(t), 0 < p_1 I \leq P(t) \leq p_2 I,\ \forall t \geq 0$*, which satisfies the equation*

$$A(t)^\top P(t+1) A(t) - P(t) = -Q(t) \tag{19}$$

with $Q(t)$ *continuous, symmetric, positive definite matrix,* $Q(t) \geq q_1 I > 0$*, then the equilibrium point* $x=0$ *is globally exponentially stable.*

Proof: *The Lyapunov function $V(t,x) = x^\top P(t)x$ satisfies*

$$p_1\|x\|^2 \le V(t,x) \le p_2\|x\|^2$$

Moreover, consider the absolute difference

$$\begin{aligned} V(t+1, A(t)x) - V(t,x) &= x^\top A(t)^\top P(t+1)A(t)x - x^\top P(t)x \\ &= x^\top [A(t)^\top P(t+1)A(t) - P(t)]x = -x^\top Q(t)x \end{aligned}$$

Therefore it holds

$$V(t+1, A(t)x) - V(t,x) \le -q_1\|x\|^2$$

and the assumptions of theorem 5.7 are satisfied. □

Define now the transition matrix $\Phi(t, t0)$ which is such that $x(t) = \Phi(t,t_0)x(t_0)$. It holds $\Phi(t_0,t_0) = I$ and for linear time variant system its form is $\Phi(t,t_0) = A(t-1)\cdot A(t-2)\cdots A(t_0)$.

Theorem 5.10 (Condition on the transition matrix to have exp stab) *The equilibrium point $x = 0$ of the linear time variant system*

$$x(t+1) = A(t)x(t)$$

is uniformly asymptotically stable if and only if the state transition matrix satisfies

$$\|\Phi(t,t_0)\| \le ke^{-\lambda(t-t_0)},\ \forall\ t \ge t_0 \ge 0 \tag{20}$$

for some positive constants k and λ.

Proof: *We first introduce 2 lemmas*

Lemma 5.11 *If system (18) is uniformly stable, then there exists a constant M independent of t_0 such that $\|\Phi(t,t_0)\| \le M$ for all $t \ge t_0$.*

Proof: If the system is stable, fixing ϵ, we can choose $\delta > 0$ such that if $\|x(t_0)\| \le \delta$, then $\|\Phi(t,t_0)x(t_0)\| \le \epsilon$ for all $t \ge t_0$. It follows

$$\max_{\|x\|=\delta} \|\Phi(t,t_0)x\| = \max_{\|x\|=1} \|\Phi(t,t_0)\delta x\| = \delta \max_{\|x\|=1} \|\Phi(t,t_0)x\| \le \epsilon$$

so, using the induced norm,

$$\|\Phi(t,t_0)\| = \max_{\|x\|=1} \|\Phi(t,t_0)x\| \le \frac{\epsilon}{\delta^{-1}} := M$$

□

Lemma 5.12 *The following statements are equivalent*

(i) *System (18) is uniformally asymptotically stable*

(ii) *System (18) is globally uniformly asymptotically stable*

(iii) $\|\Phi(t,t_0)\| \to 0$ *as* $t \to \infty$ *uniformly in* t_0

(iv) *Given* $\{z_i\}_{i=1}^n$ *a basis of* R^n, *then* $\|\Phi(t,t_0)z_i\| \to 0$ *as* $t \to \infty$ *uniformly in* t_0

Proof: *(i)* $\Rightarrow$ *(ii) follows from linearity, and the implications (ii)* $\Rightarrow$ *(iv) and (iii)* $\Rightarrow$ *(i) can be easily verified. Concerning (iv)* $\Rightarrow$ *(iii) we can proceed as follows: if (iv) holds, then for every* $\epsilon > 0$ *there exists a time* $\tau(\epsilon)$ *independent of* t_0 *such that* $\|\Phi(t,t_0)z^i\| < \epsilon$ *for all* $t \geq t_0 + \tau(\epsilon)$ *and* $i = 1,\dots,n$. *For every* $x(t_0) = \sum_{i=1}^n \xi_i z_i$, $\|x(t_0)\| = 1$, *there exists a positive constant* a *such that* $\max|\xi_i| \leq a^{-1}$. *Thus*

$$\|\Phi(t,t_0)x(t_0)\| = \left\|\sum_{i=1}^n \xi_i \Phi(t,t_0)z_i\right\| \leq a^{-1}n\epsilon,\ t \geq t_0 + \tau(\epsilon)$$

and again, due to the induced norm, this proves (iii). □

Going back to the proof of the theorem, if the transition matrix satisfies (20), then due to Lemma 5.12 the system is uniformly asymptotically stable. On the other hand suppose that the system is uniformly asymptotically stable. By Lemma 5.12, there exists $\tau \geq 0$ *such that* $\|\Phi(t+\tau,t)\| \leq 1/2$ *for all* $t \geq t_0$. *It follows*

$$\|\Phi(t_0+k\tau,t_0)\| \leq \|\Phi(t_0+k\tau,t_0+(k-1)\tau)\| \dots \|\Phi(t_0+\tau,t_0)\| \leq 2^{-k}$$

Now suppose $t_0 + k\tau \leq t < t_0 + (k+1)\tau,\ t \geq t_0,\ k \in \mathbb{N}$, *then*

$$\|\Phi(t,t_0)\| \leq \|\Phi(t,t_0+k\tau)\|\|\Phi(t_0+k\tau,t_0)\| \leq \|\Phi(t,t_0+k\tau)\|2^{-k}$$

Now, due to Lemma 5.11, there exists a constant M' *such that* $\|\Phi(t,t_0+k\tau)\| \leq M'$ *for all* $t \geq t_0 + k\tau,\ k \in \mathbb{N}$ *and so*

$$\|\Phi(t,t_0)\| \leq M' 2^{-[(t-t_0)/\tau - 1]},\ t \geq t_0$$

Choosing $k = 2M'$ *and* $\lambda = -1/(\tau)\log_e(2)$ *the theorem is proved.* □

This theorem show that uniform asymptotic stability is equivalent to exponential stability.

Theorem 5.13 (An exp stable lin syst has a Lyap funct) *Let* $x = 0$ *be the exponentially stable equilibrium point of the linear time variant system*

$$x(t+1) = A(t)x(t)$$

and suppose that $A(t)$ *is bounded. Let* $Q(t)$ *be a bounded, positive definite, symmetric matrix, i.e.* $0 < q_1 I \leq Q(t) \leq q_2 I$. *Then, there is a bounded, positive definite, symmetric matrix* $P(t)$, *i.e.* $0 < p_1 I \leq P(t) \leq p_2 I$, *that satisfies (19). Hence* $V(t,x) = x^\top P(t)x$ *is a Lyapunov function for the system, that also satisfies the conditions of Theorem 5.7.*

Proof: *Let*

$$P(t) = \sum_{\tau=t}^{\infty} \Phi(\tau,t)^\top Q(\tau)\Phi(\tau,t)$$

Therefore we have

$$V(t,x) = x^\top P(t)x = \sum_{\tau=t}^{\infty} x^\top \Phi(\tau,t)^\top Q(\tau)\Phi(\tau,t)x \leq q_2 \sum_{\tau} \|\Phi(\tau,t)x\|^2$$

Using theorem 5.10, we have

$$V(t,x) \leq q_2\|x\|^2 \sum_{\tau=t}^{\infty} k^2 e^{-2\lambda(\tau-t)} = \frac{q_2 k^2}{1-e^{-2\lambda}}\|x\|^2 \leq p_1\|x\|^2$$

On the other hand

$$V(t,x) \geq q_1 \sum_{\tau=t}^{\infty} x^\top \Phi(\tau,t)^\top \Phi(\tau,t)x \geq q_1\|x\|^2$$

considering only the first element of the summation. So we have

$$q_1\|x\|^2 \leq V(t,x) \leq p_1\|x\|^2 \Rightarrow q_1 I \leq P(t) \leq p_1 I$$

and so $P(t)$ is positive definite and bounded.
Now let us check whether (19) is satisfied, so let us evaluate

$$A(t)^\top P(t+1)A(t) - P(t) = A(t)^\top \sum_{\tau=t+1}^{\infty} \left[\Phi(\tau,t+1)^\top Q(\tau)\Phi(\tau,t+1)\right] A(t) - \\ - \sum_{\tau=t}^{\infty} \Phi(\tau,t)^\top Q(\tau)\Phi(\tau,t)$$

Now since $\Phi(\tau,t+1)A(t) = \Phi(\tau,t)$, it holds

$$\sum_{\tau=t+1}^{\infty} \Phi(\tau,t)^\top Q(\tau)\Phi(\tau,t) - \sum_{\tau=t}^{\infty} \Phi(\tau,t)^\top Q(\tau)\Phi(\tau,t) = -\Phi(t,t)^\top Q(t)\Phi(t,t) = -Q(t)$$

so (19) is satisfied. From the latter we have

$$V(t+1,A(t)x) - V(t,x) = -x^\top Q(t)x \leq -q_1\|x\|^2$$

and so $V(t,x)$ satisfies all the assumptions of Theorem 5.7 □

Now, using the Lyapunov function for the linear system we will prove some linearisation results. Consider again the general nonlinear nonautonomous system

$$x(t+1) = f(t,x)$$

where $f\colon \mathbb{T} \times D \to \mathbb{R}^n$ is locally Lipschitz in x on $\mathbb{T} \times D$, and $D = \{x \in \mathbb{R}^n \mid \|x\| < r\}$. Suppose that $f(t,0) = 0$, $\forall t \in \mathbb{T}$, that is $x = 0$ is an equilibrium point for the system. Moreover suppose that the Jacobian matrix $[\partial f/\partial x]$ is bounded and Lipschitz on D, from which it follows, for all $i = 1,\ldots,n$

$$\left\|\frac{\partial f_i}{\partial x}(t,x_1) - \frac{\partial f_i}{\partial x}(t,x_2)\right\|_2 \leq L_1\|x_1 - x_2\|_2,\ \forall x_1,x_2 \in D,\ \forall t \in \mathbb{T}$$

By the mean value theorem, there exists a $z_i \in D$ on the line segment between the origin and $x \in D$ such that

$$f_i(t,x) = f_i(t,0) + \frac{\partial f_i}{\partial x}(t,z_i)x$$

Since $f(t,0)=0$, $f_i(t,x)$ can be rewritten as

$$f_i(t,x) = \frac{\partial f_i}{\partial x}(t,0)x + \underbrace{\left[\frac{\partial f_i}{\partial x}(t,z_1) - \frac{\partial f_i}{\partial x}(t,0)\right]x}_{g_i(t,x)}$$

Defining $A(t) = \frac{\partial f}{\partial x}(t,0)$, $f(t,x)$ can be rewritten as

$$f(t,x) = A(t)x + g(t,x) \tag{21}$$

The nonlinear part is bounded in norm, since

$$\begin{aligned}\|g(t,x)\|_2 &\le \left(\sum_{i=1}^{n} \left\| \frac{\partial f_i}{\partial x}(t,z_1) - \frac{\partial f_i}{\partial x}(t,0)\right\|_2^2\right)^{1/2} \|x\|_2 \le \\ &\le \left(\sum_{i=1}^{n} L_1^2 \underbrace{\|z_i\|^2}_{\le \|x\|^2}\right)^{1/2} \|x\|_2 \le \underbrace{\sqrt{n}L_1}_{L} \|x\|_2^2 \end{aligned} \tag{22}$$

This implies that in a neighbourhood of the origin we can approximate the nonlinear function $f(t,x)$ with its linearisation $A(t)x$. We can therefore apply the Lyapunov function found for the linearised system to the starting nonlinear system.

Theorem 5.14 (If lin system is exp stable than the nonlin is exp stabl) *Let $x=0$ be an equilibrium point for the nonlinear system*

$$x(t+1) = f(t,x)$$

where $f\colon \mathbb{T}\times D \to \mathbb{R}^n$ is locally Lipschitz in x on $\mathbb{T}\times D$, and $D = \{x \in \mathbb{R}^n \,|\|x\| < r\}$. Suppose that the Jacobian matrix $[\frac{\partial f}{\partial x}]$ is bounded and Lipschitz on D, uniformly in t. Let

$$A(t) = \left.\frac{\partial f}{\partial x}(t,x)\right|_{x=0}$$

Then the origin is an exponentially stable equilibrium point for the nonlinear system if it is an exponentially stable equilibrium point for the linear system $x(t+1) = A(t)x(t)$.

Proof: *From the assumptions we have that $\|A(t)\| \le B_A$. Due to theorem 5.13, given bounded and positive definite matrices $Q(t)$, $t \in \mathbb{T}$ there exist bounded and positive definite matrices matrices $P(t)$ such that $V(t,x) = x^\top P(t)x$ is a Lyapunov function for the linearised system system. The matrices $P(t)$ and $Q(t)$ satisfy the following inequalities*

$$0 < p_1 I \le P \le p_2, \quad 0 < q_1 I \le Q \le q_2 I$$

Let us use function $V(x,t)$ for the nonlinear system. To prove that it is a Lyapunov function also for the nonlinear system we have to check whether the *absolute difference $V(t+1, f(t,x)) - V(t,x)$ is negative definite. Using the rewriting (21) for function f, we have the following*

$$
\begin{aligned}
&V(t+1, f(t,x)) - V(t,x) = (x^\top A(t)^\top + g(t,x)^\top)P(t+1)(A(t)x + g(t,x)) - x^\top P(t)x = \\
&\quad = x^\top (A(t)^\top P(t+1)A(t) - P(t))x + 2g(x,t)^\top P(t+1)A(t)x + g(t,x)^\top P(t+1)g(t,x) = \\
&\quad = x^\top (-Q(t))x + 2g(x,t)^\top P(t+1)A(t)x + g(t,x)^\top P(t+1)g(t,x) \leq \\
&\quad \leq -q_1\|x\|_2^2 + 2p_1LB_A\|x\|_2^3 + p_1L^2\|x\|_2^4 = (-q_1 + 2p_1LB_A\|x\|_2 + p_1L^2\|x\|_2^2)\|x\|_2^2
\end{aligned}
$$

For the latter to be negative definite, the term $-q_1 + 2p_1LB_A\|x\|_2 + p_1L^2\|x\|_2^2$ has to be negative. As in the autonomous case, this is a parabola directed upward and with the vertex in the third quarter, so there exists $\bar{\delta} > 0$ such that as long as $\|x\| = \bar{\delta}$ then $V(t+1, f(t,x)) - V(t,x) = 0$. Choosing $\delta < \bar{\delta}$, $\delta < r$, and defining the set $B_\delta = \{x | \|x\| \leq \delta\}$, if $x \in B_\delta$ then $V(t,x)$ is a Lyapunov function for the nonlinear function. □

Corollary 5.15 *If the assumptions of Theorem 5.14 are satisfied, there exists a Lyapunov function $V(t,x)$ for the nonlinear system defined in $\mathbb{T} \times B_\delta$ that satisfies the following inequalities*

$$
\begin{gathered}
c_1\|x\|^2 \leq V(t,x) \leq c_2\|x\|^2 \\
V(t+1, f(t,x)) - V(t,x) \leq -c_3\|x\|^2 \\
|V(t,x) - V(t,y)| \leq c_4\|x-y\|(\|x\| + \|y\|)
\end{gathered}
$$

for all $x, y \in B_\delta$ and for some positive constants c_1, c_2, c_3 and c_4.

Proof: *Due to the assumptions of the Theorem 5.14, the nonlinear system is exponentially stable for $x(t_0) \in B_\delta$, so the assumptions of Theorem 5.8 are satisfied.* □

6 Convergence results for a particular class of nonlinear systems

Consider the system

$$
\begin{cases}
x(k+1) = x(k) + \epsilon\phi(k; x(k), \xi(k)) \\
\xi(k+1) = \varphi(k; \xi(k), x(k))
\end{cases}
\tag{23}
$$

For a given $\bar{k} \in \mathbb{N}$ consider also the system

$$
\tilde{\xi}(k+1; x(\bar{k})) = \varphi\left(k; \tilde{\xi}(k; x(\bar{k})), x(\bar{k})\right) \tag{24}
$$

for $k \geq \bar{k}$. Assume that there exist $\xi^*\left(k; x(\bar{k})\right)$ such that the quantity

$$
\tilde{\xi}'(k; x(\bar{k})) := \tilde{\xi}(k; x(\bar{k})) - \xi^*\left(k; x(\bar{k})\right) \tag{25}
$$

satisfies the property

$$
\|\tilde{\xi}'(k; x(\bar{k}))\| \leq C\rho^{k-\bar{k}}\ \|\tilde{\xi}'(\bar{k}; x(\bar{k}))\|. \tag{26}
$$

We have the following proposition.

Proposition 6.1 *Given $x(\bar{k})$, consider the evolution of system* (24) *for $k \geq \bar{k}$. Assume property* (26) *holds true. Then there exists a function W with the following properties:*

(i) there exist positive constants a_1 and a_2, $a_1 \leq a_2$, such that

$$a_1 \|\tilde{\xi}'(k; x(\bar{k}))\|^2 \leq W(k; \tilde{\xi}'(k; x(\bar{k})), x(\bar{k})) \leq a_2 \|\tilde{\xi}'(k; x(\bar{k}))\|^2; \quad (27)$$

(ii) there exists a constant a_3 such that

$$W(k+1; \tilde{\xi}'(k+1; x(\bar{k})), x(\bar{k})) - W(k; \tilde{\xi}'(k; x(\bar{k})), x(\bar{k})) \leq a_3 \|\tilde{\xi}'(k; x(\bar{k}))\|^2; \quad (28)$$

(iii) there exists a constant a_4 such that

$$|W(k; \tilde{\xi}'_1, x) - W(k; \tilde{\xi}'_2, x)| \leq a_4 \|\tilde{\xi}'_1 - \tilde{\xi}'_2\| \left(\|\tilde{\xi}'_1\| + \|\tilde{\xi}'_2\| \right); \quad (29)$$

(iv) there exists a constant a_5 such that

$$|W(k; \tilde{\xi}', x_1) - W(k; \tilde{\xi}', x_2)| \leq a_5 \|\tilde{\xi}'\|^2 \|x_1 - x_2\|. \quad (30)$$

Proof: The proof follows from standard Lyapunov arguments. □

Before proceeding with the proof, it is useful to give the following Propositions

Proposition 6.2 *Let $x = 0$ be an equilibrium point for the nonlinear system*

$$x(t+1) = f(t; x(t)) \quad (31)$$

where f is continuosly differentiable on $D = \{x \in \mathbb{R}^n \mid \|x\| < r\}$ and the Jacobian matrix $[\partial f/\partial x]$ is bounded and Lipschitz on D, uniformly in t. Let

$$A(t) = \frac{\partial f}{\partial x}(t; x)|_{x=0}.$$

Then, the origin is an exponentially stable equilibrium point for the nonlinear system if it is exponentially stable equilibrium point for the linear system

$$x(t+1) = A(t)x(t).$$

The following result follows from the above proposition.

Corollary 6.3 *Consider system* (31) *and assume assumptions of Proposition 6.2 hold true. Then there exist a suitable Lyapunov function V and a n-th dimensional ball B^n_r with the following properties*

(i) There exist positive constants a_1 and a_2 such that

$$a_1 \|x\|^2 \leq V(k; x) \leq a_2 \|x\|^2 \quad (32)$$

for all k and for all $x \in B^n_r$;

(ii) There exist positive constants a_3 *and* ϵ *such that*

$$V\left(k+1; x(k+1)\right) - V\left(k; x(k)\right) \leq -\epsilon a_3 \|x(k)\|^2 \tag{33}$$

for all $x(k) \in B_r^n$*;*

(iii) There exists a positive constant a_4 *such that*

$$|V(k; x_1) - V(k; x_2)| \leq a_4 \|x_1 - x_2\| (\|x_1\| + \|x_2\|) \tag{34}$$

for all x_1 *and* x_2 *in* B_n^r*.*

Now let $\xi'(k)$ be defined as

$$\xi'(k) := \xi(k) - \xi^*(k; x(k))$$

Observe that $\xi'(k) = \tilde{\xi}'(k; x(k))$.

We can write that

$$\begin{aligned}
\xi'(k+1) &= \xi(k+1) - \xi^*(k+1; x(k+1)) \\
&= \varphi(k; \xi(k), x(k)) - \xi^*(k+1; x(k+1)) \\
&= \varphi(k; \xi(k), x(k)) - \xi^*\left(k+1; x(k) + \epsilon\phi(k; x(k), \xi(k))\right) \\
&= \varphi\left(k; \xi'(k) + \xi^*(k; x(k)), x(k)\right) - \xi^*\left(k+1, x(k) + \epsilon\phi(k; x(k), \xi'(k) + \xi^*(k; x(k)))\right)
\end{aligned}$$

Now we analyze the system

$$\begin{cases} x(k+1) = x(k) + \epsilon\phi(k; x(k), \xi'(k) + \xi^*(k; x(k))) \\ \xi'(k+1) = \varphi\left(k; \xi'(k) + \xi^*(k; x(k)), x(k)\right) - \xi^*\left(k+1, x(k) + \epsilon\phi(k; x(k), \xi'(k) + \xi^*(k; x(k)))\right) \end{cases} \tag{35}$$

The following Proposition characterize the convergence properties of system 23.

Proposition 6.4 *Consider 23. Assume that properties 25 and 26 hold true. Assume system*

$$x(t+1) = x(t) + \epsilon\phi\left(k; x(k), \xi^*(k; x(k))\right)$$

satisfies assumptions of Proposition 6.2 and let $r > 0$ *be such that for* $x \in B_r^n$*, there exists a Lyapunov function as stated in Corollary 6.3. Assume function* $\phi(k; \cdot, \cdot)$ *and* $\xi^*(k; \cdot)$ *are Lipschitz with the respect to* $x \in B_r^n$ *uniformly in* k*.*

Then, there exists ϵ^* *such that for all* $\epsilon \in (0, \epsilon^*]$ *the trajectory* $x(t)$ *converges exponentially to* 0*, i.e., there exist* $C > 0$ *and* $0 < \lambda < 1$ *such that*

$$\|x(t)\| \leq C\lambda^t \|x(0)\|$$

if $x(0) \in B_r^n$*.*

Proof: *Let*

$$\chi(k+1) = x(k) + \epsilon\phi\left(k; x(k), \xi^*(k; x(k))\right) \tag{36}$$

so that $\chi(k+1) = x(k+1)$ *as soon as* $\xi'(k) = 0$*.*

The following basic bounds follow immediately from the Lipschitz and vanishing properties of the various functions

$$\|\phi\left(k; x(k), \xi^*(k; x(k))\right)\| \leq \ell_1 \|x(k)\| \tag{37}$$
$$\|\phi\left(k; x(k), \xi'(k) + \xi^*(k; x(k))\right)\| \leq \ell_2 \left(\|x(k)\| + \|\xi'(k)\|\right) \tag{38}$$
$$\|x(k+1) - x(k)\| \leq \epsilon \ell_2 \left(\|x(k)\| + \|\xi'(k)\|\right) \tag{39}$$
$$\|\phi\left(k; x(k), \xi'(k) + \xi^*(k; x(k))\right) - \phi\left(k; x(k), \xi^*(k; x(k))\right)\| \leq \ell_3 \|\xi'(k)\| \tag{40}$$
$$\|x(k+1) - \chi(k+1)\| \leq \epsilon \ell_3 \|\xi'(k)\| \tag{41}$$
$$\|\varphi(k; x(k), \xi'(k) + \xi^*(k; x(k))) - \xi^*(k; x(k))\| \leq \ell_4 \left(\|x(k)\| + \|\xi'(k)\|\right) \tag{42}$$
$$\|\xi^*(k+1; x(k+1))) - \xi^*(k; x(k)))\| \leq \ell_5 \left(\|x(k)\| + \|\xi'(k)\|\right) \tag{43}$$
$$\|x(k+1)\| \leq \ell_6 \left(\|x(k)\| + \|\xi'(k)\|\right) \tag{44}$$
$$\|\chi(k+1)\| \leq \ell_7 \left(\|x(k)\| + \|\xi'(k)\|\right) \tag{45}$$
$$\|\xi^*(k; x(k))\| \leq \ell_8 \|x(k)\| + c_r \tag{46}$$
$$\|\xi^*(k+1; x(k+1))) - \xi^*(k+1; x(k)))\| \leq \ell_9 \left(\|x(k+1) - x(k)\|\right) \tag{47}$$

for some positive constants $\ell_1, \ldots, \ell_8$ *and where* c_r *is a suitable constant depending on* r*.*

Observe now that there exist a Lyapunov function V *such that*

(i) There exist positive constants c_1 *and* c_2 *such that*

$$c_1 \|x\|^2 \leq V(k; x(k)) \leq c_2 \|x\|^2 \tag{48}$$

(ii) There exist positive constants c_3 *and* ϵ *such that*

$$V\left(k+1; \chi(k+1)\right) - V\left(k; x(k)\right) \leq -\epsilon c_3 \|x(k)\|^2 \tag{49}$$

(iii) There exists a positive constant c_4 *such that*

$$|V(k; x_1) - V(k; x_2)| \leq c_4 \|x_1 - x_2\| (\|x_1\| + \|x_2\|) \tag{50}$$

As for the temporal evolution of $V(k; x)$*, exploiting the definition* (36) *and properties* (49) *and* (50) *it follows that*

$$\begin{aligned}\Delta V(k; x(k)) &:= V(k+1; x(k+1)) - V(k; x(k)) \\ &= V(k+1; x(k+1)) - V(k+1; \chi(k+1)) + V(k+1; \chi(k+1)) - V(k; x) \\ &\leq c_4 \|x(k+1) - \chi(k+1)\| \left(\|x(k+1)\| + \|\chi(k+1)\|\right) - \epsilon c_3 \|x(k)\|^2\end{aligned}$$

and, thus using properties (41), (44) *and* (45),

$$\Delta V(k, x(k)) \leq \epsilon c_4 \ell_3 \|\xi'\| \left(\ell_6 (\|x(k)\| + \|\xi'(k)\|) + \ell_7 \|\xi'(k)\|\right) - \epsilon c_3 \|x(k)\|^2$$

Letting then $\ell_9 = c_4 \ell 3 \ell_6$, $\ell_1 0 = c_4 \ell_3 (\ell_6 + \ell_7)$ *we obtain the quadratic bound*

$$\Delta V(k, x(k)) \leq -\epsilon c_3 \|x(k)\|^2 + \epsilon 2 \ell_{10} \|x(k)\| \|\xi'(k)\| + \epsilon \ell_{11} \|\xi'(k)\|^2 \tag{51}$$

Observe that there is a Lyapunov function $W(k;x(k),\xi'(k))$ such that

$$b_1\|\xi'(k)\|^2 \le W\left(k;\xi'(k),x(k)\right) \le b_2\|\xi'(k)\|^2 \tag{52}$$

$$W\left(k+1;\varphi\left(k;\xi'(k)+\xi^*(k,x(k)),x(k)\right)-\xi^*(k+1,x(k)),x(k)\right)-W\left(k;\xi'(k),x(k)\right) \le -b_3\|\xi'(k)\|^2 \tag{53}$$

$$|W(k;\xi_1',x)-W(k;\xi_2',x)| \le b_4\|\xi_1'-\xi_2'\|\left(\|\xi_1'\|+\|\xi_2'\|\right) \tag{54}$$

$$|W(k;\xi',x_1)-W(k;\xi',x_2)| \le b_5\,\|\xi'\|^2\,\|x_1-x_2\| \tag{55}$$

Now let us compute

$$\begin{aligned}
&\Delta W\left(k;\xi'(k),x(k)\right)\\
&= W\left(k+1;\xi'(k+1),x(k+1)\right)-W\left(k;\xi'(k),x(k)\right)\\
&= W\left(k+1;\varphi\left(k;\xi'(k)+\xi^*\left(k;x(k)\right),x(k)\right)-\xi^*\left(k+1;x(k)+\epsilon\phi\left(k;x(k),\xi(k)\right)\right),x(k+1)\right)-\\
&\quad -W\left(k;\xi'(k),x(k)\right)\\
&= W\left(k+1;\varphi\left(k;\xi'(k)+\xi^*\left(k;x(k)\right),x(k)\right)-\xi^*\left(k+1;x(k)+\epsilon\phi\left(k;x(k),\xi(k)\right)\right),x(k+1)\right)\\
&\qquad -W\left(k+1;\varphi\left(k;\xi'(k)+\xi^*\left(k;x(k)\right),x(k)\right)-\xi^*\left(k+1;x(k)\right),x(k+1)\right)\\
&\qquad +W\left(k+1;\varphi\left(k;\xi'(k)+\xi^*\left(k;x(k)\right),x(k)\right)-\xi^*\left(k+1;x(k)\right),x(k+1)\right)\\
&\qquad -W\left(k+1;\varphi\left(k;\xi'(k)+\xi^*\left(k;x(k)\right),x(k)\right)-\xi^*\left(k+1;x(k)\right),x(k)\right)\\
&\qquad +W\left(k+1;\varphi\left(k;\xi'(k)+\xi^*\left(k;x(k)\right),x(k)\right)-\xi^*\left(k+1;x(k)\right),x(k)\right)\\
&\qquad -W\left(k;\xi'(k),x(k)\right)
\end{aligned}$$

We then exploit: (54) *to bound the first two rows of the last right hand side;* (55) *to bound the third and the fourth rows;* (53) *to bound the last two rows. This implies that* $\Delta W\left(k;\xi'(k),x(k)\right) \le \beta_1+\beta_2+\beta_3$*, where the last three symbols are the following shorthands:*

$$\begin{aligned}
\beta_1 &= b_4\|\xi^*\left(k+1;x(k+1)\right)-\xi^*\left(k+1;x(k)\right)\|\,\big(\|\varphi\left(k;\xi'(k)+\xi^*\left(k;x(k)\right),x(k)\right)-\xi^*\left(k+1;x(k+1)\right)\|+\\
&\qquad +\|\varphi\left(k;\xi'(k)+\xi^*\left(k;x(k)\right),x(k)\right)-\xi^*\left(k+1;x(k)\right)\|\big)\\
\beta_2 &= b_5\|\varphi\left(k;\xi'(k)+\xi^*\left(k;x(k)\right),x(k)\right)-\xi^*\left(k;x(k)\right)\|^2\|x(k+1)-x(k)\|\\
\beta_3 &= -b_3\|\xi'(k)\|^2
\end{aligned}$$

To bound β_1 *we apply the triangular inequality so that*

$$\begin{aligned}
\beta_1 &\le b_4\|\xi^*\left(k+1;x(k+1)\right)-\xi^*\left(k+1;x(k)\right)\|\,\big(\|\varphi\left(k;\xi'(k)+\xi^*\left(k;x(k)\right),x(k)\right)-\xi^*\left(k+1;x(k+1)\right)\|+\\
&\qquad +\|\xi^*\left(k+1;x(k+1)\right)-\xi^*\left(k+1;x(k)\right)\|\\
&\qquad +\|\varphi\left(k;\xi'(k)+\xi^*\left(k;x(k)\right),x(k)\right)-\xi^*\left(k+1;x(k+1)\right)\|\big)\\
&\le b_4\|\xi^*\left(k+1;x(k+1)\right)-\xi^*\left(k+1;x(k)\right)\|\,\big(2\,\|\varphi\left(k;\xi'(k)+\xi^*\left(k;x(k)\right),x(k)\right)-\xi^*\left(k+1;x(k+1)\right)\|+\\
&\qquad +\|\xi^*\left(k+1;x(k+1)\right)-\xi^*\left(k+1;x(k)\right)\|\big)\\
&\le b_4\|\xi^*\left(k+1;x(k+1)\right)-\xi^*\left(k+1;x(k)\right)\|\,\big(2\,\|\varphi\left(k;\xi'(k)+\xi^*\left(k;x(k)\right),x(k)\right)-\xi^*\left(k;x(k)\right)\|+\\
&\qquad +2\,\|\xi^*\left(k+1;x(k+1)\right)-\xi^*\left(k;x(k)\right)\|+\|\xi^*\left(k+1;x(k+1)\right)-\xi^*\left(k+1;x(k)\right)\|\big)
\end{aligned}$$

Exploiting (42), (43), (47) *and* (39), *we can write that*

$$\begin{aligned}
\beta_1 &\le \epsilon b_4\ell_2\ell_9\left(\|x(k)\|+\|\xi'(k)\|\right)\left(2\ell_4\left(\|x(k)\|+\|\xi'(k)\|\right)+\epsilon\ell_5\left(\|x(k)\|+\|\xi'(k)\|\right)\right)\\
&\le \epsilon b_4\ell_2\ell_9\left(2\ell_4+\epsilon\ell_5\right)\left(\|x(k)\|+\|\xi'(k)\|\right)^2
\end{aligned}$$

Concerning β_2, consider that from $\|x(k)\| \leq r'$, and $\|\xi(k)\| \leq r_0$ for a suitable r' and r_0, we obtain

$$\begin{aligned}\beta_2 &\leq \epsilon b_5 \ell_2 \ell_4^2 \left(\|x(k)\| + \|\xi'(k)\|\right)^2 \left(\|x(k)\| + \|\xi'(k)\|\right) \\ &\leq \epsilon b_5 \ell_2 \ell_4^2 \left(r_0 + (\ell_8 + 1)r' + c_r\right) \left(\|x(k)\| + \|\xi'(k)\|\right)^2 .\end{aligned}$$

Given the previous, we can thus write

$$\Delta W(k; x(k), \xi'(k)) \leq (\epsilon\ell_{12} + \epsilon^2\ell_{13} - b_3)\|\xi'(k)\|^2 + 2\left(\epsilon\ell_{12} + \epsilon^2\ell_{13}\right)\|x(k)\|\|\xi'(k)\| + (\epsilon\ell_{12} + \epsilon^2\ell_{13})\|x(k)\|^2 \tag{56}$$

for suitable constants ℓ_{12}, ℓ_{13}.

Now we propose a Lyapunov function for the whole system. Let the candidate be

$$U(k; x(k), \xi'(k)) = V(k; x(k)) + W(k; x(k), \xi'(k))$$

We must check whether, for all plausible trajectories, the condition $(x(k), \xi'(k)) \neq (0, 0)$ implies

$$\Delta U(k; x(k), \xi'(k)) = U(k+1; x(k+1), \xi'(k+1)) - U(k; x(k); \xi'(k)) < 0.$$

Consider that inequalities (51) and (56) form a quadratic form that can be rewritten as

$$\Delta U(k; x(k), \xi'(k)) \leq [\|x(k)\| \;\; \|\xi'(k)\|]\, A \begin{bmatrix} \|x(k)\| \\ \|\xi'(k)\| \end{bmatrix} \tag{57}$$

where

$$A = \begin{bmatrix} -\epsilon c_3 + (\epsilon\ell_{12} + \epsilon^2\ell_{13}) & \epsilon\ell_{10} + \epsilon\ell_{12} + \epsilon^2\ell_{13} \\ \epsilon\ell_{10} + \epsilon\ell_{12} + \epsilon^2\ell_{13} & \epsilon\ell_{11} + \epsilon\ell_{12} + \epsilon^2\ell_{13} - b_3 \end{bmatrix}$$

Consider now that the leading principal minors of A are, in Landau notation anf for $\epsilon \to 0$,

$$-\epsilon c_3 + O(\epsilon^2), \;\; \epsilon c_3 b_3 + O(\epsilon^2).$$

Thus there must exists a sufficiently small ϵ^* such that for every $\epsilon \in (0, \epsilon^*]$, A is negative definite, i.e.,

$$A \leq -\epsilon\ell_{14} I$$

for a suitable positive scalar ℓ_{14}.

It follows that

$$\begin{aligned}\Delta U(k; x(k), \xi'(k)) &\leq -\epsilon\ell_{14}\left(\|x(k)\|^2 + \|\xi'(k)\|^2\right) \\ &\leq -\epsilon\ell_{14}\left(\frac{1}{c_2}V(x(k)) + \frac{1}{b_2}W(k; x(k), \xi'(k))\right) \\ &\leq -\epsilon\gamma\, U(k; x(k), \xi'(k))\end{aligned}$$

where $\gamma = \ell_{14} \min\left\{\frac{1}{b_2}, \frac{1}{c_2}\right\}$. This eventually implies that

$$\begin{bmatrix} \|x(k)\| \\ \|\xi'(k)\| \end{bmatrix} \leq \ell\,(1 - \epsilon\gamma)^{\frac{k}{2}} \begin{bmatrix} \|x(0)\| \\ \|\xi'(0)\| \end{bmatrix} \tag{58}$$

where ℓ is an appropriate constant. This concludes the proof. □

7 Another convergence result for discrete time dynamical systems using time scale separation principle

Lemma 7.1 *Consider the dynamical system*

$$\begin{bmatrix} x(t+1) \\ y(t+1) \end{bmatrix} = \begin{bmatrix} I & -\epsilon B \\ C(t) & F(t) \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}. \tag{59}$$

Let the following assumptions hold

(i) There exists a matrix G such that $y = Gx$ satisfies the expression $y = C(t)x + F(t)y, \ \forall t, \forall x$

(ii) the system

$$z(t+1) = F(t)z(t) \tag{60}$$

is exponentially stable;

(iii) the system

$$\dot{x}(t) = -BGx(t) \tag{61}$$

is exponentially stable.

(iv) The matrices $C(t)$ and $F(t)$ are bounded, i.e. there exists $m > 0$ such that $\|C(t)\| < m, \|F(t)\| < m, \forall t \geq 0$.

Then, there exists $\bar{\epsilon}$, with $0 < \epsilon < \bar{\epsilon}$ such that the origin is an exponentially stable equilibrium of (59). □

Proof: *[Proof of Lemma 7.1] Let us first consider the following change of variable:*

$$z(t) = y(t) - Gx(t)$$

The dynamics of the system in the variables x, z can be written after some straightforward manipulations as follows:

$$\begin{bmatrix} x(t+1) \\ z(t+1) \end{bmatrix} = \left(\underbrace{\begin{bmatrix} I - \epsilon BG & 0 \\ 0 & F(t) \end{bmatrix}}_{\Sigma(t)} + \epsilon \underbrace{\begin{bmatrix} 0 & -BG \\ GBG & GB \end{bmatrix}}_{\Gamma} \right) \underbrace{\begin{bmatrix} x(t) \\ z(t) \end{bmatrix}}_{\mu(t)} \tag{62}$$

*where we used Assumption 1. From Assumptions 2, 3 and 4, using converse Lyapunov theorems [**?**], it follows that there exist positive definite matrices $P_x > 0$ and $P_z(t) > 0$ such that*

$$\begin{aligned} &- P_x BG - G^T B^T P_x \leq -aI, \\ &F(t)^T P_z(t+1) F(t) - P_z(t) \leq -aI, \forall t \end{aligned}$$

where a is a positive scalar and $P_z(t)$ is bounded, i.e. $\|P_z(t)\| \leq m$. We will use the following positive definite Lyapunov function to prove exponential stability *of the whole system:*

$$U(x,z,t) = x^T P_x x + z^T P_z(t) z$$
$$= \begin{bmatrix} x^T & z^T \end{bmatrix} \underbrace{\begin{bmatrix} P_x & 0 \\ 0 & P_z(t) \end{bmatrix}}_{P(t)} \begin{bmatrix} x \\ z \end{bmatrix}$$

If we define time difference of the Lyapunov function as $\Delta U(x,z,t) = U(x(t+1), z(t+1), t+1) - U(x(t,)z(t),t)$ *we get:*

$$\Delta U(x,z,t) =$$
$$x^T \left(-\epsilon(P_x BG + G^T B^T P_x) + \epsilon^2 G^T B^T P_x BG\right) x$$
$$+ z^T \left(F(t)^T P_z(t+1) F(t) - P_z(t)\right) z$$
$$+ 2\epsilon \mu^T \Sigma^T(t) P(t+1)\Gamma\mu + \epsilon^2 \mu^T \Gamma^T P(t+1)\Gamma\mu$$
$$\leq -\epsilon a\|x\|^2 - a\|z\|^2 + \epsilon^2 \underbrace{\|P_x^{\frac{1}{2}} BG\|^2}_{b} \|x\|^2$$
$$+ 2\epsilon \mu^T \Sigma^T(t) P(t+1)\Gamma\mu + \epsilon^2 \|P^{\frac{1}{2}}(t+1)\Gamma\|^2 \|\mu\|^2$$

Note that the top left block of Γ *is zero and that* $\Sigma(t)$ *and* $P(t)$ *are diagonal and bounded for all times. From this it follows that*

$$\Sigma^T(t) P(t+1)\Gamma = \begin{bmatrix} 0 & \star \\ \star & \star \end{bmatrix}$$
$$\Longrightarrow 2\mu^T \Sigma^T(t) P(t+1)\Gamma\mu \leq c(2\|x\|\|z\| + \|z\|^2)$$

for some positive scalar c*. Boundedness of* $P(t)$ *also implies that*

$$\|P^{\frac{1}{2}}(t+1)\Gamma\|^2 \|\mu\|^2 \leq d(\|x\|^2 + \|z\|^2)$$

for some positive scalar d*. Putting all together we get*

$$\Delta U(x,z,t) \leq$$
$$\begin{bmatrix} [\|x\| & \|z\|] \end{bmatrix} \begin{bmatrix} -\epsilon a + b\epsilon^2 & \epsilon c \\ \epsilon c & -a + \epsilon c + \epsilon^2 d \end{bmatrix} \begin{bmatrix} \|x\| \\ \|z\| \end{bmatrix}$$

It follows immediately that there exists a critical $\bar{\epsilon}$ *such that for* $0 < \epsilon < \bar{\epsilon}$ *the matrix in the above equation is strictly negative definite and therefore the system is exponentially stable.* □

References

[1] H. Khalil *Nonlinear systems*, Third Edition, 1996 , Prentice Hall New Jersey

[2] W. Hahn *Uber die Anwendung der Methode von Ljapunov auf Differenzengleichungen*, 1958 , Mathematische Annalen