LQG control over communication channels: the role of data losses, delays and SNR limitations

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Abstract

In this work we address the problem feedback control design in the presence of a communication channel, which gives rise to tightly coupled limitations in terms of quantization errors, decoding/computational delays and packet loss affecting the closed loop control performance. We restrict our analysis in the context of LQG control subject to SNR limitations, packet loss, and delay, and we derive their impact on optimal design for the controller parameters. In particular, we show that the stability of the closed loop system depends on a tradeoff among quantization, packet loss probability and delay. Through this analysis we are also able to recover, as special cases, several results already available in the literature that have treated packet loss, quantization error and delay separately. We also show that the estimator and controller cannot be design independently even if the controller has full knowledge of the packet loss sequence and the control inputs to the plant. In fact the optimal control gain, when accounting for the communication constraints is, in general, different from the optimal gain derived under the classical LQG scenario, which is recaptured when the SNR over the channel goes to infinity.

Key words: Control under Communication Constraints, Packet Losses, SNR Limited Channels, Delays, LQG Control, Separation Principle.

1 Introduction

Traditionally, control theory and communication theory have been developed independently and have reached considerable success in providing fundamental tools for designing information technology systems. The major objective of control theory has been to develop tools to stabilize unstable plants and to optimize some performance metrics in closed loop under the assumption that the communication channels between sensors and controller and between the controller and the plant were ideal, i.e. without distortion, packet loss or delay. This assumption actually holds in many control applications where the non idealities of the communication channel have negligible impact, compared to the effects of noise and uncertainty in the plants. With the advent of wireless communication, the Internet and the need for high performance control systems, however, the sharp separation between control and communication has been questioned and a growing body of literature has appeared from both the communication and the control communities trying to analyze the interaction between control and communication.

This recent branch of research is known as *Networked Control System* (NCS) and considers control systems wherein the control loops are closed through a real-time network, and feedback signals are exchanged in the form of data packets.

Recent results in this area have revealed the existence of a strict connection between the performance of the controlled plant and the Shannon capacity of the feedback channel. More precisely the paper [28] has introduced the concept of *anytime capacity* to characterize feedback control under communication constraints, whereas in [18] fundamental limitations on the achievable performances have been studied. Stabilization of unstable plants through a control loop has been studied for signal-to-noise ratio (SNR) limited channels [19,3,31], rate-limited [24,35,21] or lossy channels [34,14,10,16,30,13], providing links between the channel limitation (SNR, rate or loss probability) and the unstable eigenvalues of the system. Recent extensions concern the study of transient performance under feedback constraints, see e.g. [4,12], of packet loss models accounting for correlations [15,37,23,25,21] as well as model uncertainties [25].

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A subsequent step has been made to include multiple channel limitations into the model, such as packet loss and quantization [36,17], which however result in complex optimization problems. The paper [2] addresses the LQG control problem under communication constraints arising from delays and bit rate limitations, but does not account for the possibility of data losses. It shows that, if the innovation process is encoded with a finite alphabet and transmitted over a channel with distortion and delay, the separation principle still holds. Yet transmitting the innovation process requires that either the coder has full knowledge of the packet loss sequence or no losses occur. In the context of filtering the recent work [8] studies possible tradeoffs when no information on the packet loss sequence is available to the source coder.

In this work, we address the problem of performance optimization in a NCS with a communication channel model which includes SNR limitations, losses and delays. More specifically, we consider the Linear-Quadratic-Gaussian (LQG) control problem, which consists in finding the control signal of a linear time-invariant (LTI) plant that minimizes a quadratic cost function of the system state, when both the system state and the output signal are affected by Gaussian noise. While the optimal solution to the LQG problem in LTI systems with *ideal feedback channel* is known to be achieved by a controller formed by a Kalman filter and a linear-quadratic regulator, the solution to the problem in NCS systems with *realistic feedback channels* has only been investigated for specific feedback channel models, while the general solution still remains unknown.

The main contributions of this work are as follows: (i) we provide a feedback channel model that is characterized in terms of delay d, Signal-to-Noise-Ratio (SNR) ρ (and, in turn, code rate limit), and packet loss probability ϵ, while still being mathematically amenable to analysis; (ii) we setup an optimal LQG control problem and provide, for a *scalar unstable channel*, necessary and sufficient conditions for stabilizability in terms of the parameters of our channel model, and analyze the joint effect of these parameters on the system stability; (iii) as a byproduct we also obtain an expression for the system performance that can hence be used to optimize the channel parameters. Preliminary results can be found in [6,5] and [26].

We warn the reader that our channel model is intended as an abstraction of a realistic channel, but we do not enter into the important issues of coding, modulation, and decoding schemes as, for instance, is done in [20], where it is shown that the computational complexity of these operations plays a fundamental role.

The paper is articulated as follows: Section 2 contains the problem formulation and the channel modeling. In Section 3 we develop a model for scalar plants that accounts for constant delays, losses and SNR limitations, then introduces the structure of the controller and provides an explicit formulation of the optimal control problem. Section 4 presents the core results of our paper, which are compared in Section 5 against previous relevant results obtained in the literature. To illustrate the main findings of our work, we present some numerical results in Section 6. Finally, Section 7 draws conclusions. All proofs are in the Appendix.

2 Problem formulation

In this section, we cast the LQG problem into the NCS framework. First, we introduce the LQG problem and then we model the feedback transmission channel to characterize the NCS structure considered in this work. Finally, we formally define the LQG problem in the NCS architecture

2.1 LQG problem definition

We consider a plant, modeled as a discrete-time, scalar, LTI system, subject to additive white Gaussian measurement and process noise. More specifically the state of the system at step t, denoted as x_t, evolves according to the following linear model:

$$\begin{aligned} x_{t+1} &= a x_t + u_t + w_t \\ y_t &= x_t + v_t \end{aligned} \tag{1}$$

where u_t and y_t represent the input and output signals of the plant, respectively, whereas w_t and v_t are two independent discrete-time Gaussian white noise processes with variance σ_w^2 and σ_v^2, respectively. [1]

We consider the as performance index the function $J(r)$ given by the *steady state* variance of the plant output y_t, plus a control penalty, namely [2]

$$J(r) = \limsup_{t\to+\infty} \mathbb{E}[y_t^2] + r\mathbb{E}[u_t^2]\,. \tag{2}$$

The objective of the LQG problem is (i) to give conditions under which $J(r)$ can be made finite using a specific estimator-controller pair and (ii) to minimize J by properly designing such estimator and controller.

2.2 Feedback channel modeling

In the NCS framework, the plant output y_t is not directly accessible to the controller, but must be delivered through a finite capacity channel by means of a suitable transmission scheme. In Fig. 1 we provide the abstraction of a finite capacity communication scheme that includes an Additive

[1] We assume a scaling of state and input such that the state-to-output matrix is unitary, and the input is weighted with a unitary coefficient on the state update equation. This can always be done by rescaling accordingly also the weight r on the control objective (2).

[2] Of course, the term “steady state” is meaningful only if the limit is finite.

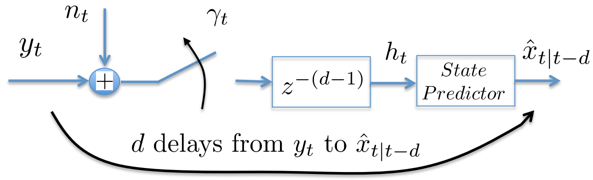

Fig. 1. Equivalent model of the feedback channel and state predictor, accounting for the presence of quantization noise, packet loss and decoding delays.

White Gaussian Noise (AWGN) n_t, an independent identically distributed (iid) Bernoulli process $\gamma_t \in \{0,1\} \sim \mathcal{B}(\epsilon)$, which models erasures/losses, and a delay of d steps between the plant output y_t and the predicted state.

The noise n_t is assumed to be independent of the signal y_t, with variance $\sigma_n^2 = \mathbb{E}\left[n_t^2\right]$ proportional to the signal variance $\sigma_y^2 = \mathbb{E}\left[y_t^2\right]$, so that the signal-to-noise ratio (SNR), $\rho = \sigma_y^2/\sigma_n^2$, can be seen as a system parameter. Thus, summing up, the feedback channel model considered in this paper has the following input-output relationship

$$h_t = \gamma_{t-d+1}(y_{t-d+1} + n_{t-d+1}), \tag{3}$$

and it is, hence, completely characterized by three parameters, namely ϵ, d, and $\alpha := 1/\rho$, with

$$d \geq 1, \quad \mathbb{P}[\gamma_t = 0] = \epsilon, \quad \sigma_n^2 = \alpha \mathbb{E}\left[y_t^2\right]. \tag{4}$$

These parameters are clearly related; see for instance [27] where a compact and tight approximation for the maximum information rate allowed for a given code length and error probability has been presented.

3 System Modeling and Structure of the Controller

In order to handle the delay in a compact form, we use the standard technique of state augmentation and define

$$\xi_t := [x_{t-d+1} \cdots x_t]^\top. \tag{5}$$

The augmented state satisfies

$$\begin{aligned} \xi_{t+1} &= A\xi_t + Bu_t + Bw_t; \\ h_t &= \gamma_{t-d+1}\left(C\xi_t + v_{t-d+1} + n_{t-d+1}\right); \\ y_t &= H\xi_t + v_t; \end{aligned} \tag{6}$$

where

$$A := \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & a \end{bmatrix}; \quad B := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}; \quad \begin{aligned} C &= [1 \ 0 \ \dots \ 0]; \\ H &= [0 \ 0 \ \dots \ 1]. \end{aligned}$$

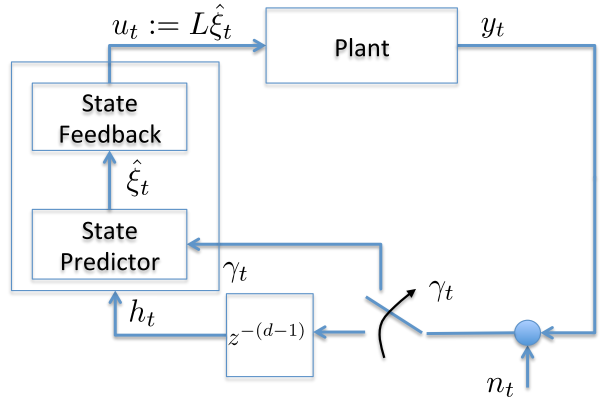

Fig. 2. NCS scheme for scalar output plants, where the plant decoder is given by the cascade of a linear state predictor and a state feedback.

We restrict our attention to the classical LQG structure for the plant decoder, which is given by the cascade of a linear state estimator and a state feedback, as represented in Fig. 2, although this might not be the optimal architecture in this context. The state estimator $\hat{\xi}_t$ (which uses the data up to time $t-d$) is governed by the following law

$$\hat{\xi}_{t+1} = A\hat{\xi}_t + Bu_t + \gamma_{t-d+1}G\left(h_t - C\hat{\xi}_t\right) \tag{7}$$

where G is a *constant* estimator gain vector, and the estimator (7) is time-varying since it depends on the sequence γ_t. In fact, if a packet is not received correctly, i.e. $\gamma_t = 0$, then the estimator updates its state using the model only, while when $\gamma_t = 1$ the estimate is adjusted by a correction term, based on the output innovation, similarly to a Kalman filter. The state feedback module, in turn, will simply return a control signal proportional to the predicted state through L, i.e.,[3]

$$u_t = L\hat{\xi}_t = [\ell_1 \ \ell_2 \ \dots \ \ell_d]\hat{\xi}_t. \tag{8}$$

This scheme was first proposed in [29] and, although it does not yield the optimal time-varying Kalman filter [34], it has the advantage of being computationally simpler and allowing for the explicit computation of the performance J, as will be shown in the next section. In this framework, the objective is to solve the following optimization problem:

$$\min_{G,L} \ J(r; G, L) \quad \text{s.t. } \mathbb{E}[||n_t||^2] = \alpha \mathbb{E}[||y_t||^2], \ \ n_t \perp y_t \tag{9}$$

where we made explicit the fact that J is a function of G and L, given the proposed control architecture. In order to fix terminology we give the following definition:

[3] Note that u_t in (8) is a function of measurements h_t up to time $t-1$, i.e. of the signal $y_s + n_s$ up to time $s = t-d$.

Definition 1 *System* (1) *is said to be* mean square stabilizable *using the control architecture in Fig. 2 if problem* (9) *admits a solution G^*, L^* that yields a finite value of the cost J for $r = 0$.* [4]

The first constraint in equation (9) implies the power of the noise n_t is always proportional to the power of the signal s_t at any time t with the same scaling factor $\alpha = 1/\rho$ allowed by the channel code rate. Later on in this paper, Theorem 2 provides the conditions under which problem (9) admits solution, while Theorem 1 shows that the optimal G^* and L^* have the following special structure:

$$G^* = [a^0 g^* \ a^1 g^* \ \ldots \ a^{d-1} g^*]^\top , \quad L^* = [0 \ 0 \ \ldots \ \ell_d] .$$

Although in this study we limit our attention to the case of scalar systems, the same mathematical machinery can be extended to the multidimensional case. We leave this generalization to future work.

3.1 Controller/Estimator model and LQG cost

As a first step, we derive the dynamical equations that govern the state as well as the error evolution for the estimator in equation (7). Inserting the control law (8) in (6) and (7) we obtain:

$$\begin{aligned} \xi_{t+1} &= A\xi_t + BL\hat{\xi}_t + Bw_t ; \\ \hat{\xi}_{t+1} &= A_L\hat{\xi}_t + \gamma_{t-d+1}G\big(C\tilde{\xi}_t + v_{t-d+1} + n_{t-d+1}\big) . \end{aligned} \tag{10}$$

where $\tilde{\xi}_t := \xi_t - \hat{\xi}_t$ and $A_L := A + BL$.

Let us now define $\bar{C} := [C \ C]$, $\bar{A} :=$ block diag$\{A_L, A\}$, $\bar{G} := [G^\top \ \ -G^\top]^\top$, $\bar{B} := [0 \ \ B^\top]^\top$ and $\bar{A}_\gamma := \bar{A} + \gamma\bar{G}[0 \ \ C]$. It follows that the equation of the feedback loop system can be written in terms of the joint state $s_t := [\hat{\xi}_t^\top, \tilde{\xi}_t^\top]^\top$ as:

$$\begin{aligned} s_{t+1} &= \bar{A}_{\gamma_{t-d+1}} s_t + \bar{B}w_t + \gamma_{t-d+1}\bar{G}\left[v_{t-d+1} + n_{t-d+1}\right] ; \\ y_{t-d+1} &= \bar{C}s_t + v_{t-d+1} . \end{aligned} \tag{11}$$

It is a well known fact that a linear system with random parameters is mean square stable if and only if the state variance $P_t := \mathrm{Var}\{s_t\}$ admits a positive semidefinite and finite limit $P := \lim_{t\to\infty} P_t$ [7], which is the unique solution of a Lyapunov-like equation that can be derived as follows. Let $\sigma_y^2 = \bar{C}P\bar{C}^\top + \sigma_v^2$ and $\sigma_n^2 = \alpha\sigma_y^2$. After some algebra, we can show that the steady state variance P satisfies the Lyapunov-type equation

$$\begin{aligned} P =& (1-\epsilon)\bar{A}_1 P\bar{A}_1^\top + \epsilon\bar{A}_0 P\bar{A}_0^\top + \bar{B}\sigma_w^2\bar{B}^\top + \\ &+ (1-\epsilon)\bar{G}\left[\sigma_v^2 + \sigma_n^2\right]\bar{G}^\top \end{aligned} \tag{12}$$

that, substituting σ_n^2 and σ_y^2 in (12), becomes

$$\begin{aligned} P =& \ (1-\epsilon)\bar{A}_1 P\bar{A}_1^\top + \epsilon\bar{A}_0 P\bar{A}_0^\top + \\ &+ \sigma_w^2\bar{B}\bar{B}^\top + \alpha(1-\epsilon)\bar{G}\bar{C}P\bar{C}^\top\bar{G}^\top \\ &+ (1-\epsilon)(1+\alpha)\bar{G}\sigma_v^2\bar{G}^\top =: \mathcal{M}(G, L, P) \end{aligned} \tag{13}$$

Remark 1 *The operator $\mathcal{M}(G, L, P)$ in* (13) *is linear in P. Conditions for existence of a fixed point (and equivalently of convergence of the instantaneous state variance where P_t replaces P on the right hand side and P_{t+1} replaces P on the left hand side) can be easily found via vectorization. Theorem 2 will provide necessary and sufficient conditions on ρ, ϵ and d for the existence of L and G that guarantee existence of such a solution.*

Minimization of the cost function (2) is equivalent to minimization of

$$J(r; G, L) = \mathbb{E}\left[\xi_t^\top C^\top C\xi_t + r\hat{\xi}_t^\top L^\top L\hat{\xi}_t\right] = \bar{C}P\bar{C}^\top + rLP_{11}L^\top . \tag{14}$$

As a simple consequence of observability of the pair (A, C) (and thus $(\bar{A}, \bar{C})$), we have the following lemma:

Lemma 1 *System* (1) *is mean square stabilizable, i.e.,* (14) *is finite for some choice of L and G, iff* (13) *admits a fixed point for some choice of L and G.*

Hence, the LQG-type optimal control problem can be written as:

$$J^*(r) := \min_{G,L} J(r; G, L) \quad s.t. \ P = \mathcal{M}(G, L, P) \tag{15}$$

and L^*, G^* will denote the optimal gains, which can be found adapting the results in [6] as explained in the next section.

4 Solution to the Optimal Control Problem

The derivation of the solution to the LQG-type optimal control problem (15), which uses results from [7], will go through the following steps:

i) We first introduce the Lagrangian

$$\begin{aligned} \mathcal{L}(P,\Lambda,L,G) := J(r;G,L) - \mathrm{Tr}\{\Lambda\,(P - \mathcal{M}(G, L, P))\} \\ s.t. \quad P = P^\top \geq 0 , \quad \Lambda = \Lambda^\top . \end{aligned} \tag{16}$$

[4] Theorem 2 shows that the existence of the solution of problem (9) for $r = 0$ is a sufficient and necessary condition for the existence of a solution for any $r > 0$. In other words, if the system (1) is mean square stabilizable, then problem (9) admits solution for any $r \geq 0$.

For future reference let us define with P_{ij} and Λ_{ij} the $n \times n$ blocks of $P \in \mathbb{R}^{2n\times 2n}$ and $\Lambda \in \mathbb{R}^{2n\times 2n}$ respectively. According to the matrix maximum principle [1] the necessary conditions for optimality of G^* and L^* are

$$\frac{\partial \mathcal{L}}{\partial P} = 0\,, \quad \frac{\partial \mathcal{L}}{\partial \Lambda} = 0\,, \quad \frac{\partial \mathcal{L}}{\partial L} = 0\,, \quad \frac{\partial \mathcal{L}}{\partial G} = 0\,. \qquad (17)$$

ii) Proposition 1 derives the necessary conditions for optimality.

iii) Theorem 1 shows that, under the assumption that the solution is unique, the gains have a special structure.

iv) Theorem 2 shows that the solution exists, it is unique for $r > 0$ and, therefore, the necessary conditions for optimality provide the solution to (15).

Proposition 1 *The necessary conditions* (17) *for stationarity of the Lagrangian* (16) *admit the solution*

$$P_{12}^* = P_{21}^* = 0\,, \quad \Lambda_{11}^* = \Lambda_{12}^* = \Lambda_{21}^*\,;$$

and

$$\begin{aligned} G^* &= A P_{22}^* C^\top \Sigma_\alpha^{-1}\,; \\ L^* &= -\left(B^\top \Lambda_{11}^* B + r\right)^{-1} B^\top \Lambda_{11}^* A\,; \end{aligned} \qquad (18)$$

where

$$\Sigma_\alpha := (1+\alpha)\left(\sigma_v^2 + C P_{22}^* C^\top\right) + \alpha C P_{11}^* C^\top\,. \qquad (19)$$

L^* and G^* are the candidate optimal gains for the LQG-type optimal control problem (15). *The matrices P_{11}^*, P_{22}^*, Λ_{11}^* and Λ_{22}^* can be found solving the following (coupled) Riccati-type equations*

$$\begin{aligned} P_{11}^* &= A_{L^*} P_{11}^* A_{L^*}^\top + (1-\epsilon) A P_{22}^* C^\top \Sigma_\alpha^{-1} C P_{22}^* A^\top\,; \\ P_{22}^* &= A P_{22}^* A^\top + \sigma_w^2 B B^\top - (1-\epsilon) A P_{22}^* C^\top \Sigma_\alpha^{-1} C P_{22}^* A^\top\,; \\ \Lambda_{11}^* &= A_{L^*}^\top \Lambda_{11}^* A_{L^*} + r\,(L^*)^\top L^* \\ &\quad + \alpha(1-\epsilon) C^\top (G^*)^\top (\Lambda_{22}^* - \Lambda_{11}^*) G^* C + C^\top C\,; \\ \Lambda_{22}^* &= \epsilon A^\top \Lambda_{22}^* A + C^\top C + \sigma_w^2 B B^\top + (1-\epsilon) A^\top \Lambda_{11}^* A + \\ &\quad (1-\epsilon)(A - G^* C)^\top (\Lambda_{22}^* - \Lambda_{11}^*)(A - G^* C) + \\ &\quad + \alpha(1-\epsilon) C^\top (G^*)^\top (\Lambda_{11}^* - \Lambda_{22}^*) G^* C\,. \end{aligned} \qquad (20)$$

where $A_{L^*} := A + BL^*$.

We now report two important results (see the Appendix for a proof) which characterize the structure, existence and uniqueness of the stabilizing estimator-controller pair (G^*, L^*). First we show that, provided it exists, the solution (G^*, L^*) have a special structure that guarantees the control algorithm can be implemented with memory equal to the state dimension.

Theorem 1 *Assume that, for suitable L and G, equation* (13) *admits a solution. Then the gains (G^*, L^*) satisfy*

$$\begin{aligned} G^* &= \left[g^* \; ag^* \; a^2 g^* \; \ldots \; a^{d-1} g^*\right]^\top; \\ L^* &= \left[0 \; 0 \; \ldots \; \ell_d\right], \quad \ell_d \in \mathbb{R}\,. \end{aligned} \qquad (21)$$

If $r = 0$ then

$$L^* = \left[0 \; 0 \; \ldots \; -a\right], \qquad (22)$$

so that A_{L^*} is nilpotent, i.e. the optimal controller is deadbeat.

We now show that the cost $J(r; G, L)$ is finite only provided a certain relation between packet loss probability ϵ, SNR ρ, and delay d is satisfied. This condition neatly extends the well known condition for the zero delay case [6]. Under the same condition the solution is also unique.

Theorem 2 *The optimal control problem* (9) *(or equivalently* (15)*) admits a solution if and only if*

$$\delta := \frac{1-\epsilon}{1+\alpha a^{2d}} > 1 - \frac{1}{a^2} \qquad (23)$$

If $r > 0$ the optimal control problem (9) *admits a unique solution, which necessarily coincides with the pair L^*, G^* in* (18). *For $r = 0$ any solution to* (15) *will correspond to the same value of the cost $J^*(0)$ which can be expressed as*

$$J^*(0) = a^{2d} p_{22}^* + \sum_{i=0}^{d-1} a^{2i} \sigma_w^2 + \sigma_v^2$$

where p_{22}^* is unique the positive solution of the scalar Modified Algebraic Riccati Equation (MARE)

$$p_{22}^* = a^2 p_{22}^* + \sigma_w^2 - \delta \frac{a^2 (p_{22}^*)^2}{p_{22}^* + \bar{r}(d)}\,; \qquad (24)$$

and

$$\bar{r}(d) := \left(1 + \alpha a^{2d}\right)^{-1} \left((1+\alpha)\,\sigma_v^2 + \alpha \sum_{i=0}^{d-1} a^{2i} \sigma_w^2\right).$$

Although the theorem has been formally derived for $d \geq 1$, it provides the correct solution also for $d = 0$, i.e., in the zero delay case that was derived in our previous work [26].

5 Discussion of the results and related literature

The previous theorem recovers some of the results available in the literature as special cases. To see this, let us preliminarily observe that (23) is equivalent to the following conditions:

$$\frac{1}{\alpha} = \rho \geq \frac{(a^2-1)a^{2d}}{1-\epsilon a^2}\,; \qquad\qquad \epsilon < \frac{1}{a^2}\,. \tag{25}$$

If we set $\rho = \infty$ ($\alpha = 0$), which is equivalent to consider a channel with infinite capacity, the first condition in (25) is always satisfied, and therefore the necessary and sufficient condition reduces to the second condition in (25) which is the same stability condition in the lossy network literature [34]. Also, it shows that in the infinite capacity scenario, the stability is independent of the delay d, as shown in [30].
Alternatively if we assume no packet loss in the channel, i.e. $\epsilon = 0$ so that the second equation in (25) is automatically satisfied, and no delay, i.e. $d = 0$, then the stability condition which stems from the first of (25) leads to

$$\rho > a^2 - 1\,, \tag{26}$$

which is the same stability condition presented in the context of SNR-limited control systems in [3]. Recalling that the channel capacity $\mathcal{C}$ cannot be lower than the source rate R_{q} that, in turn, depends logarithmically on ρ as $R_{\mathrm{q}} = \frac{1}{2}\log(1+\rho)$, we get that equation (26) is equivalent to the well known logarithmic condition

$$\mathcal{C} > \log|a|\,. \tag{27}$$

Consider now the results in [3], Theorem III.2, applied to the simplified scenario of a scalar plant $G(z) = \frac{1}{z-a}$ with unstable pole a and a channel delay $d = 1$. Consider an augmented system which is used to model the delay, i.e., $G_1(z) = \frac{1}{z^2-az}$, and apply Theorem III.2 in [3] thus obtaining the condition $\rho > a^4 - a^2$. Clearly, this postulates that the packet loss probability satisfies $\epsilon = 0$. The same result can be readily obtained from condition (23) by setting $\epsilon = 0$ and $d = 1$. On the other hand, the results in [3] cannot be generalized to incorporate the packet loss probability ϵ.
Consider now the paper [32] where a second order equivalent model for the loss is introduced in Lemma 9. However this second order equivalent formulation has non-trivial implications from the point of view of control/estimation. Under the reasonable assumption that the receiver can recognize a packet erasure, it can proceed in "prediction" mode. Instead, the modeling adopted by [32] implicitly assumes that the receiver treats the received signal (which is conventionally set to zero in the loss event, see Fig. 3a in [32] where the loss corresponds to the event $\theta = 0$) as a finite variance additive noise that overlaps to the signal y even when the data is lost, which of course would give rise to a biased estimate and degradation of performance. Incidentally, this model is very similar to that proposed in [9,11], which therefore suffers from the same problem. This is reflected, in fact, by an apparently looser stability condition than the one found in our paper for a scalar system with the same unstable pole a, signal to quantization noise ratio ρ, packet loss probability ϵ and delay $d \geq 1$. In fact, by casting our scheme into the model used in [32], we obtain an equivalent SNR which simultaneously takes into account SNR and packet loss, given by

$$\rho' = \frac{\rho(1-\epsilon)}{(1+\epsilon\rho)}\,. \tag{28}$$

If we now replace this equivalent SNR ρ' in the stability condition (23) and let $\rho \to \infty$, so that $\rho' \to (1-\epsilon)/\epsilon$, we obtain the same stability condition found in [32], namely

$$\frac{1-\epsilon}{1+\epsilon(a^{2d}-1)} > 1 - \frac{1}{a^2}\,. \tag{29}$$

Conversely, by taking the limit $\rho \to \infty$ in our stability condition (23) we get $1 - \epsilon > 1 - \frac{1}{a^2}$ that is independent of the delay and it is less stringent that condition (29) derived in [32], since $1 + \epsilon(a^{2d}-1) > 1$ for $d \geq 1$.
Consider now the paper [33]. The results found in Section 4 are recovered from the equations above only when $d = 0$. In fact, substituting ρ' defined in (28) in place of ρ in (23), setting $d = 0$, and rearranging the terms, we obtain the stability condition

$$1 - \epsilon > \left(1 + \frac{1}{\rho}\right)\left(1 - \frac{1}{a^2}\right)\,;$$

which is identical to condition (29) in [33].
Note, however, that the same does not hold for $d \geq 1$ and, as discussed above, the conditions derived in [33] are more stringent than those given in (25). As a result, this implies that the estimator/controller design provided in our paper is able to stabilize a wider class of closed loop systems under the same channel limitations when compared with the controllers proposed in [32,33].
Let us next consider the results obtained in [22]. In particular, we focus on equation (1) in [22], reported below for convenience:

$$\mathbb{E}\left[\left(\frac{a^2}{2^{2R_t}}\right)^n\right] < 1\,. \tag{30}$$

This is a necessary and sufficient condition for a scalar plant with pole a to be second order stable when controlled over a time varying channel where the instantaneous rate R_t is a random variable. Now, when the delay d is equal to zero, our channel model is captured by this framework setting $n = 1$ (blocks of length one), and $R_t = \bar{R}\gamma_t$ where γ_t are iid Bernoulli random variables, taking value 0 with probability ϵ which model the erasure events. Assuming the rate $\bar{R}$ is related to the SNR by $\bar{R} = \frac{1}{2}\log(1+\rho)$, and substituting this expression in (30), we get the condition

$$\frac{a^2}{1+\rho}(1-\epsilon) + a^2\epsilon < 1$$

that can be shown to be equivalent to our condition (23). Note however that condition (30) (i.e. condition (1) in [22]) does not account for possible delays nor, in our opinion, it can be easily modified to do so; therefore, also from this point of view, our work nontrivially extends previous results. Finally, one might be tempted to reduce any system with delay $d > 1$ into an equivalent system with delay $d = 1$ by downsampling the system with sample period d. This is equivalent to substituting $a \leftarrow a^d$ and $d \leftarrow 1$ into condition (23) which leads to

$$\frac{1-\epsilon}{1+\alpha a^{2d}} > 1 - \frac{1}{a^{2d}} .$$

While the left hand side is unchanged, the right hand side gives rise to a more stringent stability condition, thus showing the benefit of controlling the system at a higher rate than the delay. As a side note, we observe that our procedure is constructive, thus yielding a practical and very simple mean to design the controller as well as the coding/decoding scheme.

6 Numerical results

In this section we present a few numerical results that illustrate our derivation. As regards the control strategy, we observe that the separation principle (i.e., the possibility of designing separately the optimal controller and the optimal estimator) does not hold for this specific control scheme. This is illustrated in Figure 3 that shows the closed-loop pole $a_\ell := a + b\ell_d$ as a function of the control penalty coefficient r, which is varied from a very small value (virtually equal to zero) to 10^4. The upper curve with dot markers refers to the "standard" LQG control design, which assumes a control channel with infinite capacity (i.e., $\rho = \infty$), while the other curves are obtained for different values of the SNR ρ. For all curves, we set $a = 1.3$, delay $d = 4$, and $\epsilon = 0.1$. Note that, for the *cheap* control case, the optimal control is dead beat. We observe that, for a finite ρ, the gap with the standard LQG design increases with the control penalty r, while it reduces, as expected, when $\rho \to \infty$. Although it is well known that separation principle does not hold when the controller has not full knowledge of the packet loss sequence or of the control input sequence entering the plant [16,30], this is a somewhat counterintuitive example showing that quantization noise leads to the loss of separation principle even if the controller has full access to the packet loss and plant control input history.

Next, Figure 4 illustrates how the delay affects the control performance when all other parameters are fixed. In particular we consider $a = 1.5$, $\epsilon = 0.1$, $\rho = 100$ and $r = 10$. From (25), the maximum delay for which the system is stabilized is

$$d_{max} = \left\lfloor \frac{\log\left(\frac{\rho(1-\epsilon a^2)}{a^2-1}\right)}{2\log(a)} \right\rfloor = 5.$$

We have verified numerically that, indeed, if we set $d = 6$ the cost blows up.

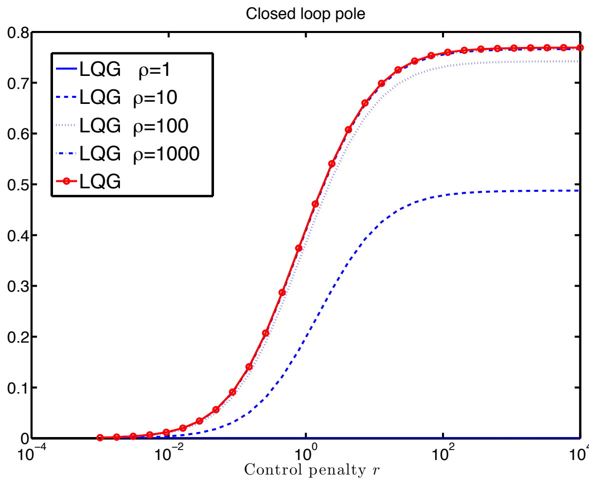

Fig. 3. "Closed loop" matrix $a_\ell := a + b\ell_d$ as a function of the control penally r. The red curve with circles corresponds to the standard LQG design without communication constraints.

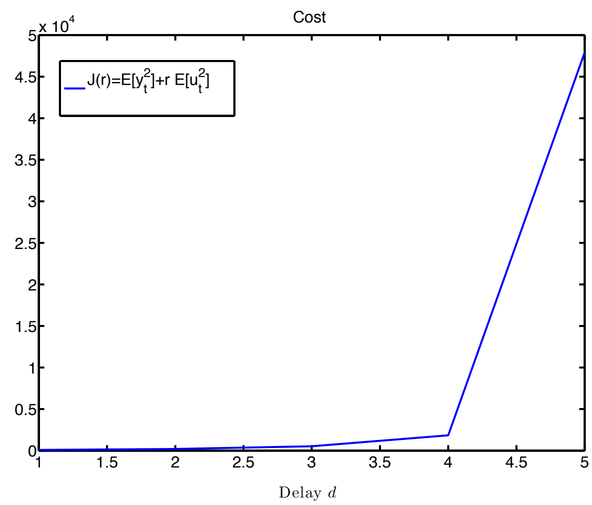

Fig. 4. Optimal cost $J^*(r) = \mathbb{E}\left[y_t^2\right] + r\mathbb{E}\left[u_t^2\right]$ as a function of delay d.

7 Conclusions and future work

We have considered an LQG control problem that accounts for code rate limitations, as well as for packet drops and delays arising from a communication channel between the sensor and the controller. We have argued in fact that there is a tight connection between the actual rate at which one can transmit information, the decoding delay and the packet-drop probability.

We have restricted our attention to a specific control architecture in which the plant outputs are transmitted via a rate limited channel and then processed through the cascade of a state estimator followed by a linear (state) feedback controller. We have considered a scalar model, with feedback channel subject to delay, packet losses, and limited transmit rate. Conditions for stability are derived in terms of a modified algebraic Riccati equation and recapture results from the literature as special cases.

The tools and methods we have developed can be used to deal with an arbitrary multivariate model; yet we have not been able, so far, to develop simple and transparent conditions for stability. As such, in order to keep the exposition as simple as possible, we have preferred to stick to the scalar case. A detailed analysis of the multivariable case is postponed to future work; some preliminary results are found in [6]. Future work will also include a detailed study of the tradeoff between packet loss, delay and SNR for specific limited capacity channels with realistic coding schemes.

Appendix

7.1 Proof of Proposition 1

Defining $\bar{L} := [L \quad 0]$, $\bar{Q} := \sigma_w^2 \bar{B}\bar{B}^\top$ and recalling that $\sigma_y^2 = \bar{C}P\bar{C}^\top + \sigma_v^2$ the Lagrangian in (16) can be written explicitly as:

$$
\begin{aligned}
\mathcal{L}(P,\Lambda,L,G) =& \mathrm{Tr}\left\{P\bar{C}^\top\bar{C}\right\} + r\mathrm{Tr}\left\{\bar{L}P\bar{L}^\top\right\} - \mathrm{Tr}\left\{P\Lambda\right\} \\
& \mathrm{Tr}\left\{\left[(1-\epsilon)\bar{A}_1 P\bar{A}_1^\top + \epsilon\bar{A}_0 P\bar{A}_0^\top + \bar{Q}\right]\Lambda\right\} \\
& + \mathrm{Tr}\left\{(1-\epsilon)\bar{G}(\sigma_v^2 + \alpha\sigma_y^2)\bar{G}^\top\Lambda\right\}
\end{aligned}
$$

Now, we need to compute the derivatives (17). Let $\frac{\partial\mathcal{F}}{\partial\Theta_{ij}}$ denote the partial derivative of a function $\mathcal{F}(\Theta) : \Theta \in \mathbb{R}^{m\times n} \to \mathbb{R}^{p\times q}$ w.r.t. the $(i,j)-th$ element of the matrix Θ and $\Delta\mathcal{F}(\Theta)_{ij} := \frac{\partial\mathcal{F}(\Theta)}{\partial\Theta_{ij}}$. In addition, we set: $\sigma_\alpha^2 := \sigma_v^2 + \alpha\sigma_y^2$. We hence have

$$
\begin{aligned}
\frac{\partial\mathcal{L}}{\partial G_{ij}} &\propto \mathrm{Tr}\left\{\left[\frac{\partial\bar{A}_1}{\partial G_{ij}}P\bar{A}_1^\top + \frac{\partial\bar{G}}{\partial G_{ij}}\sigma_\alpha^2\bar{G}^\top\right]\Lambda\right\} \\
&\propto \mathrm{Tr}\left\{\begin{bmatrix} 0 & \Delta G_{ij}C \\ 0 & -\Delta G_{ij}\end{bmatrix} P\bar{A}_1^\top + \Delta\bar{G}_{ij}\sigma_\alpha^2\bar{G}^\top\Lambda\right\}
\end{aligned}
$$

Now, defining $A_G := A - GC$ we obtain:

$$
\begin{aligned}
\frac{\partial\mathcal{L}}{\partial G_{ij}} &\propto \mathrm{Tr}\left\{\Delta\bar{G}_{ij}\begin{pmatrix} CP_{21} & CP_{22}\end{pmatrix}\bar{A}_1^\top\Lambda + \Delta\bar{G}_{ij}\sigma_\alpha^2\bar{G}^\top\right\} \\
&\propto \mathrm{Tr}\left\{\Delta\bar{G}_{ij}\begin{pmatrix} CP_{21}A_L^\top + CP_{22}C^\top G^\top & CP_{22}A_G^\top\end{pmatrix}\Lambda\right\} \\
&\quad + \mathrm{Tr}\left\{\Delta\bar{G}_{ij}\sigma_\alpha^2\bar{G}\Lambda\right\} \qquad (31) \\
&\propto \mathrm{Tr}\left\{\Delta G_{ij}\left[CP_{21}A_L^\top + \left(CP_{22}C^\top + \sigma_\alpha^2\right)G^\top\right]\Lambda_{11}\right\} \\
&\quad + \mathrm{Tr}\left\{\Delta G_{ij}\left(CP_{22}A_G^\top - \sigma_\alpha^2 G^\top\right)\Lambda_{21}\right\} \\
&\quad - \mathrm{Tr}\left\{\Delta G_{ij}\left[CP_{21}A_L^\top + \left(CP_{22}C^\top + \sigma_\alpha^2\right)G^\top\right]\Lambda_{12}\right\} \\
&\quad - \mathrm{Tr}\left\{\Delta G_{ij}\left(CP_{22}A_G^\top - \sigma_\alpha^2 G^\top\right)\Lambda_{22}\right\}.
\end{aligned}
$$

We shall see later on that the solution P^* satisfies $P_{12}^* = 0$ and Λ^* satisfies $\Lambda_{11}^* = \Lambda_{12}^* = \Lambda_{21}^*$, so that the latter equation reduces to

$$
\begin{aligned}
\frac{\partial\mathcal{L}}{\partial G_{ij}} &\propto \mathrm{Tr}\left\{\Delta G_{ij}\left[\left(CP_{22}^*C^\top + \sigma_\alpha^2\right)G^\top\right]\Lambda_{11}^*\right\} \\
&\quad + \mathrm{Tr}\left\{\Delta G_{ij}\left(CP_{22}^*A_G^\top - \sigma_\alpha^2 G^\top\right)\Lambda_{11}^*\right\} \\
&\quad - \mathrm{Tr}\left\{\Delta G_{ij}\left[\left(CP_{22}^*C^\top + \sigma_\alpha^2\right)G^\top\right]\Lambda_{11}^*\right\} \qquad (32) \\
&\quad - \mathrm{Tr}\left\{\Delta G_{ij}\left(CP_{22}^*A_G^\top - \sigma_\alpha^2 G^\top\right)\Lambda_{22}^*\right\} \\
&= \mathrm{Tr}\left\{\Delta G_{ij}\left[CP_{22}^*A^\top - \sigma_\alpha^2 G^\top - CP_{22}^*C^\top G^\top\right](\Lambda_{11}^* - \Lambda_{22}^*)\right\}
\end{aligned}
$$

Using (32), the equation $\frac{\partial\mathcal{L}}{\partial G_{ij}} = 0$ is clearly satisfied for:

$$
\begin{aligned}
G^* &:= AP_{22}^*C^\top\left(CP_{22}^*C^\top + \sigma_\alpha^2\right)^{-1} \qquad (33) \\
&= AP_{22}^*C^\top\left((1+\alpha)\left(\sigma_v^2 + CP_{22}^*C^\top\right) + CP_{11}^*C^\top\right)^{-1}.
\end{aligned}
$$

Note now that from the equation $P = \mathcal{M}(G,L,P)$ (see (12)) the equation for P_{12} reads as

$$
\begin{aligned}
P_{12} &= (1-\epsilon)\left[A_L P_{12}A_G^\top + GCP_{22}A_G^\top - G\sigma_\alpha^2 G^\top\right] \qquad (34) \\
&\quad + \epsilon A_L P_{12}A^\top
\end{aligned}
$$

Substituting $G = G^*$, (34) reduces to

$$
P_{12} = (1-\epsilon)A_L P_{12}A_{G^*}^\top + \epsilon A_L P_{12}A^\top, \qquad (35)
$$

which clearly admits $P_{12}^* = 0$ as a solution. We now consider the derivative w.r.t. L. First write

$$
\begin{aligned}
\mathcal{L}(P,\Lambda,L,G) &= \mathrm{Tr}\{P\bar{C}^\top\bar{C}\} + r\mathrm{Tr}\{P\bar{L}^\top\bar{L}\} - \\
&\quad + \mathrm{Tr}\left\{\Lambda P - (1-\epsilon)\bar{A}_1^\top\Lambda A_1 P - \epsilon\bar{A}_0^\top\Lambda A_0^\top P\right\} \\
&\quad + \mathrm{Tr}\left\{(1-\epsilon)\bar{G}^\top\Lambda\bar{G}\sigma_\alpha^2\right\} + \mathrm{Tr}\left\{\Lambda\bar{Q}\right\}
\end{aligned}
$$

so that

$$
\begin{aligned}
\frac{\partial\mathcal{L}}{\partial L_{ij}} &\propto \mathrm{Tr}\left\{(1-\epsilon)\frac{\partial\bar{A}_1^\top}{\partial L_{ij}}\Lambda\bar{A}_1 P + \epsilon\frac{\partial\bar{A}_0^\top}{\partial L_{ij}}\Lambda\bar{A}_0 P\right\} \\
&\quad + r\mathrm{Tr}\left\{\frac{\partial\bar{L}}{\partial L_{ij}}^\top\bar{L}P\right\}.
\end{aligned}
$$

Observe now that

$$
\frac{\partial\bar{A}_\ell^\top}{\partial L_{ij}} = \begin{bmatrix}\Delta L_{ij}^\top B^\top & 0 \\ 0 & 0\end{bmatrix}, \quad \ell = 0,1;
$$

and, therefore, the necessary conditions for optimality $\frac{\partial\mathcal{L}}{\partial L_{ij}} = 0$ can be written as

$$
\frac{\partial\mathcal{L}}{\partial L_{ij}} \propto \mathrm{Tr}\left\{\Delta L_{ij}^\top\left(B^\top\Lambda_{11}A_L + rL\right)P_{11}\right\} = 0. \qquad (36)
$$

Since $A_L = A + BL$ than $B^\top\Lambda_{11}A_L = B^\top\Lambda_{11}A + B^\top\Lambda_{11}BL$, therefore the previous equation returns, as its unique solution:

$$
L^* = -\left(B^\top\Lambda_{11}^*B + r\right)^{-1}B^\top\Lambda_{11}^*A. \qquad (37)
$$

It now remains to verify that the solution Λ^* satisfies $\Lambda^*_{11} = \Lambda^*_{12} = \Lambda^*_{21}$. The equation for Λ can be obtained by equating to zero the partial derivative $\frac{\partial \mathcal{L}}{\partial P_{ij}}$. After some simple calculations one obtains:

$$\Lambda = (1-\epsilon)\left(\bar{A}_1^\top \Lambda \bar{A}_1 + \alpha \bar{C}^\top \bar{G}^\top \Lambda \bar{G}\bar{C}\right) + \epsilon \bar{A}_0^\top \Lambda \bar{A}_0 + \bar{C}^\top \bar{C} \tag{38}$$

which can be expanded yielding

$$\begin{aligned}
\Lambda &= (1-\epsilon)\begin{bmatrix} A_L^\top \Lambda_{11} A_L & A_L^\top(\Lambda_{11}GC + \Lambda_{12}A_G) \\ * & * \end{bmatrix} + \\
&+\epsilon \begin{bmatrix} A_L^\top \Lambda_{11} A_L & A_L^\top \Lambda_{12} A \\ A^\top \Lambda_{21} A_L & A^\top \Lambda_{22} A \end{bmatrix} + \bar{C}^\top \bar{C} + r\bar{L}^\top \bar{L} + \\
&+\alpha(1-\epsilon)\bar{C}^\top \left[G^\top(\Lambda_{11} - \Lambda_{21} - \Lambda_{12} + \Lambda_{22})G\right]\bar{C} \\
&= (1-\epsilon)\begin{bmatrix} A_L^\top \Lambda_{11} A_L + rL^\top L & A_L^\top(\Lambda_{11}GC + \Lambda_{12}A_G) \\ * & * \end{bmatrix} + \\
&+\epsilon \begin{bmatrix} A_L^\top \Lambda_{11} A_L + rL^\top L & A_L^\top \Lambda_{12} A \\ A^\top \Lambda_{21} A_L & A^\top \Lambda_{22} A \end{bmatrix} + \bar{C}^\top \bar{C} + \\
&+\alpha(1-\epsilon)\bar{C}^\top \left[G^\top(\Lambda_{11} - \Lambda_{21} - \Lambda_{12} + \Lambda_{22})G\right]\bar{C}
\end{aligned} \tag{39}$$

where the blocks denoted with $*$ are unimportant to our purposes. Using now the expressions for L^* in (37) we observe that, for $L = L^*$, it holds $A_L^\top \Lambda^*_{11} BL + rL^\top L = 0$ (see also (36)), so that

$$A_{L^*}^\top \Lambda^*_{11} A_L + rL^\top L = A_L^\top \Lambda^*_{11} A$$

and therefore (39) can be rewritten as:

$$\begin{aligned}
\Lambda &= \epsilon \begin{bmatrix} A_L^\top \Lambda_{11} A & A_L^\top \Lambda_{12} A \\ A^\top \Lambda_{21} A_L & A^\top \Lambda_{22} A \end{bmatrix} + \bar{C}^\top \bar{C} + \\
&= (1-\epsilon)\begin{bmatrix} A_L^\top \Lambda_{11} A & A_L^\top(\Lambda_{11} - \Lambda_{12})GC + A_L^\top \Lambda_{12} A \\ * & * \end{bmatrix} + \\
&+\alpha(1-\epsilon)\bar{C}^\top \left[G^\top(\Lambda_{11} - \Lambda_{21} - \Lambda_{12} + \Lambda_{22})G\right]\bar{C} = \\
&= \begin{bmatrix} A_L^\top \Lambda_{11} A & (1-\epsilon)A_L^\top(\Lambda_{11} - \Lambda_{12})GC + A_L^\top \Lambda_{12} A \\ * & * \end{bmatrix} + \\
&+\alpha(1-\epsilon)\bar{C}^\top \left[G^\top(\Lambda_{11} - \Lambda_{21} - \Lambda_{12} + \Lambda_{22})G\right]\bar{C} + \bar{C}^\top \bar{C}
\end{aligned} \tag{40}$$

Note now that setting $\Lambda_{11} = \Lambda_{12} = \Lambda_{21}$, equation (40) becomes

$$\begin{aligned}
\begin{bmatrix} \Lambda_{11} & \Lambda_{11} \\ \Lambda_{11} & \Lambda_{22} \end{bmatrix} &= \begin{bmatrix} A_L^\top \Lambda_{11} A & A_L^\top \Lambda_{11} A \\ * & * \end{bmatrix} + \bar{C}^\top \bar{C} + \\
&+\alpha(1-\epsilon)\bar{C}^\top \left[G^\top(\Lambda_{22} - \Lambda_{11})G\right]\bar{C}
\end{aligned} \tag{41}$$

from which it is clear that $\Lambda_{11} = \Lambda_{12} = \Lambda_{21}$ is a solution. Eq. (39) just says that the off-diagonal blocks do not add constraints, so that only the diagonal blocks need to be solved for determining Λ_{11} and Λ_{22}. The off-diagonal blocks are automatically satisfied.

7.2 *Proof of Theorem 1*

We start by noting that according to the theorem hypotheses (i) there exist L and G such that the closed loop system (11) is mean square stable (i.e., (13) admits a finite solution), and also that (ii) $r > 0$, (iii) $\sigma_v^2 > 0$, (iv) and both $(A^\top, BB^\top)$, $(A, C^\top C)$ are mean square detectable. These conditions satisfy the hypotheses of Theorem 3 in [7] which guarantees that a solution P^*_{11}, P^*_{22}, Λ^*_{11}, Λ^*_{22} exists, is unique and can be obtained as fixed points of the iterates:

$$\begin{aligned}
G^i &= AP^i_{22}C^\top \Sigma_\alpha^{-1} \\
L^i &= -\left(B^\top \Lambda^i_{11} B + r\right)^{-1} B^\top \Lambda^i_{11} A \\
P^{i+1}_{11} &= A_{L^i} P^i_{11} A_{L^i}^\top + (1-\epsilon)AP^i_{22}C^\top \Sigma_\alpha^{-1} CP^i_{22}A^\top \\
P^{i+1}_{22} &= AP^i_{22}A^\top + \sigma_w^2 BB^\top - (1-\epsilon)AP^i_{22}C^\top \Sigma_\alpha^{-1} CP^i_{22}A^\top \\
\Lambda^{i+1}_{11} &= A_{L^i}^\top \Lambda^i_{11} A_{L^i} + r\left(L^i\right)^\top L^i + \\
&\quad +\alpha(1-\epsilon)C^\top (G^i)^\top (\Lambda^i_{22} - \Lambda^i_{11})G^i C + C^\top C \\
\Lambda^{i+1}_{22} &= \epsilon A^\top \Lambda^i_{22} A + C^\top C + \sigma_w^2 BB^\top + (1-\epsilon)A^\top \Lambda^i_{11} A + \\
&\quad (1-\epsilon)(A - G^i C)^\top (\Lambda^i_{22} - \Lambda^i_{11})(A - G^i C) + \\
&\quad +\alpha(1-\epsilon)C^\top (G^i)^\top (\Lambda^i_{11} - \Lambda^i_{22})G^i C
\end{aligned} \tag{42}$$

with initial conditions $P^0_{11} = P^0_{22} = \Lambda^0_{11} = \Lambda^0_{22} = 0$. Now, observe that $B^\top = [0 \dots 0 \;\; 1]$. This implies that $B^\top \Lambda^0_{11} = [0 \dots 0 \;\; \Lambda^0_{11}(d,d)]$ so that $L^0 = [0 \dots 0 \;\; \ell^0_d]$ where $\ell^0_d = -a\frac{\Lambda^0_{11}(d,d)}{\Lambda^0_{11}(d,d)+r}$. Since Λ^0_{11} is diagonal, also $A_{L^0}^\top \Lambda^0_{11} A_{L^0}$ is diagonal and Λ^1_{11} is still diagonal. As such $B^\top \Lambda^1_{11} = [0 \dots 0 \;\; \Lambda^1_{11}(d,d)]$ and therefore $L^1 = [0 \dots 0 \;\; \ell^0_d]$ where $\ell^1_d = -a\frac{\Lambda^1_{11}(d,d)}{\Lambda^1_{11}(d,d)+r}$. The same argument can be iterated showing that, provided $L^i = [0 \dots 0 \;\; *]$ and Λ^i_{11} is diagonal, then Λ^{i+1}_{11} is diagonal and $L^{i+1} = [0 \dots 0 \;\; -a\frac{\Lambda^{i+1}_{11}(d,d)}{\Lambda^{i+1}_{11}(d,d)+r}]$. Therefore, by induction, $L^i = [0 \dots 0 \;\; *] \;\; \forall i$, which implies that $L^* = \lim_{i\to\infty} L^i = [0 \dots 0 \;\; \ell^*_d]$, $\ell^*_d = -a\frac{\Lambda^*_{11}(d,d)}{\Lambda^*_{11}(d,d)+r}$. This completes the proof as far as the structure of L^* is concerned. For $r = 0$ the hypotheses of Theorem 3 in [7] are not satisfied and therefore uniqueness of the solution cannot be guaranteed. However, if we now take the limit as $r \to 0$, by continuity of the cost w.r.t. r, we obtain that for $r = 0$ one optimal solution is given by[5] $L^* = [0 \dots 0 \;\; -a]$ and thus the controller is dead-beat. Let us now consider the optimal gain G^*. From Proposition 1 we know that $\hat{\xi}_t$ and $\tilde{\xi}_t$ are uncorrelated. Therefore $\hat{\xi}_t$ can be interpreted as the projection of ξ_t on a certain stationaryred subspace Ξ_{t-d} of

[5] Note that $\Lambda^* > 0$ for all $r \geq 0$ and for all $\epsilon \in [0,1]$ since the pairs $(\bar{A}_\gamma, \bar{C}^\top), \gamma = \{0,1\}$ are reachable (see eq. (38)). Therefore also $\Lambda^*_{11}(d,d) > 0$ for all $r \geq 0$.

(the space spanned by the components of) z^{-}_{t-d+1}, i.e.,

$$\hat{\xi}_t = \hat{E}[\xi_t|\Xi_{t-d}] \quad \hat{\xi}_{t+1} = \hat{E}[\xi_{t+1}|\Xi_{t-d+1}]\,.$$

where $\hat{E}[\cdot|\cdot]$ denotes the orthogonal projection (linear minimum variance estimator). Recalling (5), we now compute the projection components of $\xi_{t+1} := [x_{t-d+2}\ x_{t-d+3}\dots x_{t+1}]$ assuming $\gamma_{t-d+1} = 1$. Using the standard Kalman measurements update and (7) if follows that

$$\hat{E}[x_{t-d+2}|\Xi_{t-d+1}] \;=\; \hat{E}[x_{t-d+2}|\Xi_{t-d}] + g^* \left(h_t - C\hat{\xi}_t\right)\,,$$

for a suitable gain g^*. Similarly,

$$\begin{aligned}\hat{E}[x_{t-d+3}|\Xi_{t-d+1}] &= a\hat{E}[x_{t-d+2}|\Xi_{t-d+1}] + bu_{t-d+2}\\ &= a\hat{E}[x_{t-d+2}|\Xi_{t-d}] + bu_{t-d+2} + ag^* \left(h_t - C\hat{\xi}_t\right)\\ &= \hat{E}[x_{t-d+3}|\Xi_{t-d}] + ag^* \left(h_t - C\hat{\xi}_t\right)\,,\end{aligned} \tag{43}$$

where the third equality has been obtained using the identity

$$x_{t-d+3} = ax_{t-d+2} + bu_{t-d+2} + w_{t-d+2}\,,$$

and the fact that w_{t-d+2} is orthogonal to Ξ_{t-d}. Iterating we obtain, $\forall k \geq 3$:

$$\begin{aligned}\hat{E}[x_{t-d+k}|\Xi_{t-d+1}] &= a\hat{E}[x_{t-d+k-1}|\Xi_{t-d+1}] + bu_{t-d+k}\\ &= \hat{E}[x_{t-d+k}|\Xi_{t-d+1}] + a^k g^* \left(h_t - C\hat{\xi}_t\right)\,,\end{aligned}$$

which shows that $G^* = [g^*\ ag^* \dots a^{d-1}g^*]$.

7.3 *Proof of Theorem 2*

The proof is divided in two parts: first we consider the cheap control case, i.e., we assume that $r = 0$ in (2), (14). Then we shall show that a solution exists for $r > 0$ if and only if it exists for $r = 0$.
Consider now $r = 0$; first of all let us observe that, using (10), the state update equation can be written in the form

$$\xi_{t+1} = A_{L^*}\hat{\xi}_t + A\tilde{\xi}_t + Bw_t\,.$$

As shown in Proposition 1 the necessary conditions admit $P_{12} = 0$ as a possible solution for any value of r. Moreover, Theorem 3 in [7] guarantees that $P^*_{12} = 0$ is the unique optimal solution for $r > 0$. If we choose $P^*_{12} = 0$ then this returns a cost that is the same cost of optimal unique solution for $r \to 0$, therefore by continuity of the cost in r, then $P^*_{12} = 0$ is surely one optimal solution. When using the gains L^* and G^* the estimate $\hat{\xi}_t$ and the error $\tilde{\xi}_t$ are uncorrelated. Our aim now is to see under which conditions the cost J is finite with this choice of L^* and G^*. Obviously, if under these conditions the optimal control problem admits a solution, then the necessary conditions are also sufficient.
If a steady state exists, then

$$\begin{aligned}\Sigma^* &:= Var\{\xi_{t+1}\} = P^*_{11} + P^*_{22}\\ &= A_{L^*}P^*_{11}A^\top_{L^*} + AP^*_{22}A^\top + \sigma^2_w BB^\top\end{aligned} \tag{44}$$

Note also that Σ^* is the Toeplitz matrix built with the covariance function of x_{t-d+i} and, as such, it is constant along the diagonal. Therefore

$$C\Sigma^*C^\top = H\Sigma^*H^\top\,, \quad H := [0\;0\;\dots\;0\;1]\,.$$

Note also that $HA_{L^*} = [0\;\dots\;0]$ so that, using (44), we get

$$\begin{aligned}C\Sigma^*C^\top &= H\Sigma^*H^\top = HAP^*_{22}A^\top H^\top + \sigma^2_w HBB^\top H^\top\\ &= [0\dots0\;a]P^*_{22}[0\dots0\;a]^\top + \sigma^2_w\\ &= a^2P^*_{22}(d,d) + \sigma^2_w\,;\end{aligned}$$

where $P^*_{22}(d,d)$ is the diagonal element in position (d,d) of the matrix P^*_{22} (the south-east corner). Now, using the fact that $P^*_{22}(i,i)$ is the $i-$steps ahead state prediction error, it is easy to see that

$$P^*_{22}(d,d) = a^{2d-2}P^*_{22}(1,1) + \sum_{i=0}^{d-2} a^{2i}\sigma^2_w\,, \tag{45}$$

so that

$$\begin{aligned}C\left(P^*_{11} + P^*_{22}\right)C^\top &= C\Sigma^*C^\top\\ &= a^2P^*_{22}(d,d) + \sigma^2_w\\ &= a^{2d}P^*_{22}(1,1) + \textstyle\sum_{i=0}^{d-1} a^{2i}\sigma^2_w\\ &= a^{2d}CP^*_{22}C + \bar{q}(d)\,,\end{aligned} \tag{46}$$

where $\bar{q}(d) := \sum_{i=0}^{d-1} a^{2i}\sigma^2_w$. We can use this last condition to manipulate Σ_α in (19) as follows:

$$\begin{aligned}\Sigma_\alpha &= (1+\alpha)\left(\sigma^2_v + CP^*_{22}C^\top\right) + \alpha CP^*_{11}C^\top\\ &= (1+\alpha)\,\sigma^2_v + CP^*_{22}C^\top + \alpha C\left(P^*_{11} + P^*_{22}\right)C^\top\\ &= \left(1+\alpha a^{2d}\right)CP^*_{22}C^\top + (1+\alpha)\,\sigma^2_v + \alpha\bar{q}(d)\\ &= \left(1+\alpha a^{2d}\right)\left(CP^*_{22}C^\top + \bar{r}(d)\right)\,,\end{aligned}$$

where the last equation defines $\bar{r}(d)$. Therefore the equation for P^*_{22} in (20) takes the form of a Modified Algebraic Riccati Equation (MARE) [34]

$$\begin{aligned}P^*_{22} &= AP^*_{22}A^\top + \sigma^2_w BB^\top\\ &\quad -\delta AP^*_{22}C^\top\left(CP^*_{22}C^\top + \bar{r}(d)\right)^{-1}CP^*_{22}A^\top\,,\end{aligned} \tag{47}$$

where

$$\delta := \frac{1-\epsilon}{1+\alpha a^{2d}}\,.$$

Note also that $HPC^\top = \mathbb{E}[\tilde{x}_{t-d+1}\tilde{x}_t] = a^{d-1}\mathbb{E}\left[\tilde{x}_{t-d+1}^2\right] = a^{d-1}P_{22}^*(1,1)$. Now using the fact that $HA = [0\ldots 0\ a]$ and multiplying (47) by H and $H^\top$ from left and right respectively, we obtain

$$HP_{22}^*H^\top = a^2HP_{22}^*H^\top + \sigma_w^2 - \delta\frac{a^2HP_{22}^*C^\top CP_{22}^*H^\top}{CP_{22}^*C^\top+\bar{r}(d)}\,. \tag{48}$$

Defining $p_{22}^* := CP_{22}^*C^\top$ and using (45), so that $HP_{22}^*H^\top = a^{2d-2}p_{22}^* + \sum_{i=0}^{d-2}a^{2i}\sigma_w^2$ equation (48) can be manipulated to yield:

$$p_{22}^* = a^2p_{22}^* + \sigma_w^2 - \delta\frac{\left(a^2p_{22}^*\right)^2}{p_{22}^*+\bar{r}(d)}\,. \tag{49}$$

It is well known (see [30]) that (49) admits a solution if and only if

$$\delta = \frac{1-\epsilon}{1+\alpha a^{2d}} > 1 - \frac{1}{a^2}$$

which yields (23). Using now (46) we immediately obtain an expression for the optimal cost:

$$\begin{aligned} J^* &= C(P_{11}^* + P_{22}^*)C^\top + \sigma_v^2 \\ &= a^{2d}CP_{22}^*C + \bar{q}(d) + \sigma_v^2 \\ &= a^{2d}p_{22}^* + \textstyle\sum_{i=0}^{d-1}a^{2i}\sigma_w^2 + \sigma_v^2\,. \end{aligned}$$

This concludes the proof for the case $r = 0$. Now, denote with $P_{ii}^*(r)$ the solutions of the coupled Riccati equations (20) as a function of r. Using the fact that $C(P_{11} + P_{22})C^\top = H(P_{11}+P_{22})H^\top$ and $L^*(r) = \ell_*(r)H$ for some $\ell_*(r) < +\infty$, we can derive the following upper bound for the optimal value of the cost $J^*(r)$ where the dependency on r is made explicit:

$$\begin{aligned} J^*(r) &= CP_{11}^*(r)C^\top + CP_{22}^*(r)C^\top + \\ &\quad +rL^*(r)P_{11}^*(r)\left(L^*(r)\right)^\top \\ &\le CP_{11}^*(r)C^\top + CP_{22}^*(r)C^\top + \\ &\quad +rL^*(r)\left(P_{11}^*(r) + P_{22}^*(r)\right)\left(L^*(r)\right)^\top \\ &= (1 + r\ell_*^2(r))H(P_{11}^*(r) + P_{22}^*(r))H^\top \end{aligned}$$

Assume now that a solution exists for $r = 0$. Clearly $J^*(0) = H(P_{11}^*(0)+P_{22}^*(0))H^\top$ is bounded and, therefore, so is $(1+r\ell_*(0)^2)H(P_{11}^*(0) + P_{22}^*(0))H^\top$. In addition the following chain of inequalities holds:

$$\begin{aligned} J^*(r) &= H(P_{11}^*(r) + P_{22}^*(r))H^\top + r\ell_*^2(r)HP_{11}^*(r)H^\top \\ &\le H(P_{11}^*(0) + P_{22}^*(0))H^\top + r\ell_*^2(0)HP_{11}^*(0)H^\top \\ &\le (1 + r\ell_*(0)^2)H(P_{11}^*(0) + P_{22}^*(0))H^\top \\ &< +\infty\,, \end{aligned}$$

where the first inequality stems from the fact that $J^*(r)$ is certainly smaller than $J(r)$ computed when G and L are chosen optimizing $J(0)$. The second inequality follows just adding $r\ell_*(0)HP_{22}(0)H^\top$. The previous equation guarantees that the optimal value $J^*(r)$ is finite and, as such, the solution exists. Conversely, if a solution exists for some $r > 0$, then $H(P_{11}^*(r) + P_{22}^*(r))H^\top \le H(P_{11}^*(r) + P_{22}^*(r))H^\top + r\ell_*^2(r)HP_{11}^*(r)H^\top = J^*(r) < +\infty$. Therefore:

$$\begin{aligned} J^*(0) &= H(P_{11}^*(0) + P_{22}^*(0))H^\top \\ &\le H(P_{11}^*(r) + P_{22}^*(r))H^\top \\ &\le (1 + r\ell_*(0)^2)H(P_{11}^*(r) + P_{22}^*(r))H^\top \\ &\le (1 + r\ell_*(0)^2)J^*(r) \\ &< +\infty \end{aligned}$$

and, therefore, a solution exists also for $r = 0$. This concludes the existence part of the proof.[6] Now, note that $\sigma_v^2 > 0$, (A, B) is a reachable pair and (A, C) is an observable pair. This last observation implies that $(A^\top, BB^\top)$ and $(A, C^\top C)$ are detectable and hence mean square detectable. If also $r > 0$, then all the assumptions of Theorem 3 in [7] are satisfied and hence the solution is unique. For $r = 0$ the theorem does not apply, yet from continuity of the cost in r, if there are multiple solutions, they will correspond the same value of cost as the one obtained using

$$L^*(0) := \lim_{r\to 0} L^*(r) \quad G^*(0) := \lim_{r\to 0} G^*(r)\,,$$

which can be therefore regarded as “the” optimal solution.

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