Multi-Robot Localization via GPS and Relative Measurements in the Presence of Asynchronous and Lossy Communication

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Abstract— **This work addresses the problem of distributed multi-agent localization in presence of heterogeneous measurements and wireless communication. The proposed algorithm integrates low precision global sensors, like GPS and compasses, with more precise relative position (i.e., range plus bearing) sensors. Global sensors are used to reconstruct the absolute position and orientation, while relative sensors are used to retrieve the shape of the formation. A fast distributed and asynchronous linear least-squares algorithm is proposed to solve an approximated version of the non-linear Maximum Likelihood problem. The algorithm is provably shown to be robust to communication losses and random delays. The use of ACK-less broadcast-based communication protocols ensures an efficient and easy implementation in real world scenarios. If the relative measurement errors are sufficiently small, we show that the algorithm attains a solution which is very close to the maximum likelihood solution. The theoretical findings and the algorithm performances are extensively tested by means of Monte-Carlo simulations.**

I. INTRODUCTION

In the last decades mobile autonomous robotics has moved out from being a small research area mainly targeting military applications to a growing business with many start-ups targeting civil applications thanks to the cost reduction in the hardware and the appearance of dedicated software for rapid prototyping. In particular, the advances in cooperative robotics using multiple vehicles have achieved results performance in controlled environments, as for example flying vehicles playing tennis [1], multiple quadrotors building sophisticated architectures [2], multiple vehicles playing multiple instruments [3], multiple vehicles transporting suspended objects [4].

Global and relative localization of the vehicles is one fundamental task that needs to be accomplished in order to accomplish many more complex tasks. Most of the remarkable results have been obtained in indoor controlled environments where location and orientation of the vehicles are typically obtained via multiple cameras that are able to track several markers on the vehicles and therefore they can rapidly estimate their exact orientation and location [5]. These estimates are computed at a central location and then forwarded to the different vehicles. Such architecture is not replicable in outdoor unstructured environments and alternative solutions have to be found. Although global position system (GPS) sensors and compass sensors are available, the accuracy they provide might be insufficient for many tasks such as tight formation control, map-building, and coordinated patrolling. As so, additional sensors that are able to measure relative position and orientation among vehicles are ought, such as stereo cameras, ultrasonic rangers, IR range finders, etc.. These sensors paired with GPS and compass could dramatically improve the accuracy of robot absolute location in outdoor settings.

Localization in unstructured environments has a long history and a whole area of research, named simultaneous localization and mapping (SLAM), has been devoted to the topic [6]. In SLAM, the main goal is to reconstruct the location and past trajectory of a vehicle based on sensory data collected along its travelling in the environment, possibly using known markers/beacons. The problem is particularly challenging since often the exact location of these beacons is not known and needs to be estimated as well. Another challenge is that estimating the location from multiple poses turns out to be a highly non-linear problem which might have multiple solutions [7], in particular if bearing-only sensors [8] or range-only sensors are used [9]. Many advances have been made when both range and bearing sensors are available, although these are often batch-based solutions where cooperation among agents is absent or communication is performed sporadically [10].

In this work we address the problem of multi-vehicle localization where the estimate has to be performed in real-time and communication is achieved via wireless communication. We propose to integrate less precise global sensors (GPS and compass) with more precise relative positioning sensors (range and bearing sensors) in order to achieve global high accuracy. Intuitively, precise range and bearing sensors would allow for the reconstruction of a relative formation but provides no information about the global position and orientation of the formation. Differently, compass and GPS installed in multiple vehicles can provide estimation of the centroid and orientation of the whole formation. The fusion of these two types of information would allow an accurate global positioning of all vehicles. Another challenge that we want to address is to provide an algorithm that is totally distributed, asynchronous and robust to communication losses. In fact, a centralized solution is not advisable in a scenario where not all vehicles can communicate with each other and a complex leader-election procedure might be needed. Moreover, synchronous communication is also difficult to enforce since it requires fine time synchronization among the different vehicles and possible packet losses might slow down the algorithm since multiple retransmissions are required to deliver the message. Although recent research has

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addressed the problem of distributed multi-robot localization from relative measurements [11]–[15], the proposed algorithms are synchronous and it is not clear how they can be employed in realistic settings when communication is lossy and asynchronous as in wireless communication.

In this work we propose an asynchronous distributed algorithm for multi-robot localization that integrates GPS, compass, range and bearing measurements and is robust to packet losses and random delays. In particular, we show that, if the range and bearing errors are sufficiently small, it is possible to linearize the localization problem achieving a performance which is very close to the exact maximum likelihood solution. Moreover, such solution can be computed via a broadcast-based communication protocols that does not require ACK packets and is therefore fast and easy to implement.

The reminder of the paper is as follows. In Section II we introduce some mathematical notation useful later on. In Section III we present the measurement model and we formulate the maximum likelihood estimator and a possible linear approximation. In Section IV we present a distributed and asynchronous solution of the problem highlighting its resilience to packet losses. Section V reports the numerical results. Finally, Section VI concludes the paper.

II. MATHEMATICAL PRELIMINARIES

Resorting to standard graph theory, the estimation problem can be naturally associated with an *undirected measurement graph* $\mathbf{G} = (\mathbf{V};\mathbf{E})$ where $\mathbf{V} \in \{1,\ldots,N\}$ represents the nodes and $\mathbf{E} \subset \mathbf{V}\times\mathbf{V}$ contains the unordered pairs of nodes $\{i,j\}$ which are connected to and measure each other. We denote with $\mathscr{N}_i \subseteq \mathbf{V}$ the set $\{j \mid \{i,j\} \in \mathbf{E}\}$, i.e. the neighboring set of node i. An undirected graph $\mathbf{G}$ is said to be connected if for any pair of vertices $\{i,j\}$ a path exists, connecting i to j. In the problem at hand, we consider a communication graph among the nodes which coincides with the measurements graph $\mathbf{G}$. Moreover, broadcast and asynchronous communications are assumed among the nodes. We denote with $|\cdot|$ the modulus of a scalar. Assuming M to be the cardinality of $\mathbf{E}$, the incidence matrix $A \in R^{M\times N}$ of $\mathbf{G}$ is defined as $A = [a_{ei}]$, where $a_{ei} = \{1,-1,0\}$, if edge e is incident on node i and directed away from it, is incident on node i and directed toward it, or is not incident on node i, respectively. We denote with the symbol $\|\cdot\|$ the vector 2-norm and with $[\cdot]^T$ the transpose operator. The symbol $\odot$ represents the *Hadamard* product. Given a vector $\mathbf{v} \in \mathbb{R}^2$, the function $\text{atan2}(\cdot) : \mathbb{R}^2 \to [0,2\pi]$ returns its angle, i.e., $\mathbf{v} = \|\mathbf{v}\| e^{j\,\text{atan2}(\mathbf{v})}$. Given a matrix $\mathbf{v} \in \mathbb{R}^{2\times n}$, with $v_{\text{ctr.}}$ we denote the vector centroid, i.e., $v_{\text{crt.}} = \frac{1}{n}\sum_{i=1}^{n} v_i$, where v_i is the i- th row of the matrix. The symbol σ_x denotes the standard deviation of the generic measurement x. The operator $\mathbb{E}[\cdot]$ denotes the expected value, while $\text{proj}(\cdot) : \mathbb{R} \to \mathbb{R}^2$ denotes the function $\text{proj}(\theta) = \begin{bmatrix}\cos\theta & \sin\theta\end{bmatrix}^T$. Finally, $\mathbb{I}$ denotes the identity matrix of suitable dimensions.

III. PROBLEM FORMULATION

Consider the problem of estimating the 2D positions, expressed in a common reference frame, of N nodes of a sensor network. Each node of the network is endowed with a set of sensors that provide both relative and absolute measurements.

In the following, firstly, we introduce the statistical models exploited for each type of measurements. Secondly, we formulate the non linear *Maximum-Likelihood* estimation problem. Thirdly, we introduce an suitable linear and convex reformulation.

A. Measurement Model

We assume that the N nodes are provided with a GPS module, a compass, a relative range sensor, and a relative bearing sensor. We denote with $p_i = (x_i, y_i),\ i \in \mathbf{V}$, the 2D position of node i in a common inertial frame, and with θ_i its orientation with respect to the inertial North axis, which in the following we assume to coincide with the x-axis. Each sensor is described by the following statistical model:

1) The GPS measurement $p_i^{\text{GPS}} = (x_i^{\text{GPS}}, y_i^{\text{GPS}})$ represents a noisy measurement of $p_i = (x_i, y_i)$. We assume a normal distribution of the GPS measurements, that is $p_i^{\text{GPS}} \sim \mathscr{N}(p_i, \sigma_p^2 \mathbb{I})$.
2) The compass provides a noisy measurement θ_i^C of θ_i. This is modelled according to an angular Gaussian distribution (see, e.g., [16]) which approximates the *Langevin* distribution [17]. This reads as $\text{proj}(\theta_i^C) \sim \mathscr{N}\left(\text{proj}(\theta_i), \sigma_\theta^2 \mathbb{I}\right)$.
3) The range sensor returns a noisy measurement r_{ij} of the distance between nodes i and j, which is modelled according to a normal distribution, that is $r_{ij} \sim \mathscr{N}(\|p_i - p_j\|, \sigma_r^2)$.
4) The bearing sensor returns a noisy measurement δ_{ij} of the bearing angle of the node j in the local frame of node i. For δ_{ij} we adopt an angular Gaussian distribution model which reads as $\text{proj}(\delta_{ij}) \sim \mathscr{N}\left(\text{proj}(\text{atan2}(p_j - p_i) - \theta_i), \sigma_\delta^2 \mathbb{I}\right)$.

Remark III.1. Observe that, in order to reduce the set-up cost, each node has access to highly noisy absolute measurements together with relative measurements that are less prone to noise than the absolute ones. In particular, the GPS sensors are usually characterized by a standard deviation $\sigma_p = 2$ [m] [18], [19], while the compass by a standard deviation $\sigma_\theta = 0.05$ [rad] [20]. To retrieve information about range and bearing different methods can be used, e.g., depth-camera, laser, ultrasound. Acceptable values for the standard deviation of these measurements might be $\sigma_r = 0.1$ [m] and $\sigma_\delta = 0.03$ [rad]. Due to the variability in the accuracy of the available sensors, we will test our algorithm in a sufficiently wide range of standard deviation values.

For the sake of simplicity, we consider that all the nodes are endowed with a GPS module. However, a simple reformulation of the problem would still guarantee that all the results hold even if a reduced number of nodes are provided with a GPS.

B. Maximum-Likelihood Estimator

We assume that all the measurements are independent and their probability distributions are given in the previous section. It is possible to formulate the localization problem as a *Maximum-Likelihood* (ML) estimation problem [21]. Let us define the state and measurements sets, respectively, as

$$\begin{aligned} \mathbf{x} &= \{\mathbf{p}, \theta\} = \{p_i,\, \theta_i \text{ with } i \in \mathbf{V}\}, \\ \mathbf{y} &= \left\{p_i^{\text{GPS}},\, \theta_i^C,\, r_{hk},\, \delta_{hk} \text{ with } i \in \mathbf{V},\ (h,k) \in \mathbf{E}\right\}, \end{aligned}$$

where $\mathbf{p} := [p_1, \ldots, p_N]^T$ and $\theta := [\theta_1, \ldots, \theta_N]^T$. Then, the negative log-likelihood cost function can be written as

$$J(\mathbf{x}) := -\log f\big(\mathbf{y}\,|\,\mathbf{x}\big) = J_p + J_\theta + J_r + J_\delta + c, \qquad (1)$$

where

$$\begin{aligned} J_p &= \sum_{i=1}^{N} \frac{\|p_i - p_i^{\text{GPS}}\|^2}{2\sigma_p^2}, \\ J_\theta &= \sum_{i=1}^{N} \frac{\left\|\text{proj}(\theta_i^C) - \text{proj}(\theta_i)\right\|^2}{2\sigma_\theta^2}, \\ J_r &= \sum_{(i,j)=1}^{M} \frac{(r_{ij} - \|p_i - p_j\|)^2}{2\sigma_r^2}, \\ J_\delta &= \sum_{(i,j)=1}^{M} \frac{\left\|\text{proj}(\delta_{ij}) - \text{proj}\left(\text{atan2}(p_j - p_i) - \theta_i\right)\right\|^2}{2\sigma_\delta^2}, \end{aligned}$$

and c is a constant term that does not depend on $\mathbf{x}$ and $\mathbf{y}$. The minimization of the function in (1) would provide the maximum-likelihood estimator for the nodes absolute positions and orientations, i.e.:

$$\widehat{\mathbf{x}}^{\text{ML}} = \text{argmin}_{\mathbf{x}}\ J(\mathbf{x}). \qquad (2)$$

The ML estimator benefits of some properties regarding its mean and its asymptotic behavior. In particular, consider the following equivalent parametrization of agents' positions using their centroid $p_{\text{ctr.}}$ and corresponding deviation Δp_i. This reads as

$$p_i = p_{\text{ctr.}} + \Delta p_i, \qquad \sum_i \Delta p_i = 0, \qquad (3)$$

Let us also define $\Delta\mathbf{p} = (\Delta p_1, \ldots, \Delta p_N)$. Thanks to the new parametrization, equation (2) is equivalent to:

$$\begin{aligned} \left\{\widehat{p}_{\text{ctr.}}^{\text{ML}}, \Delta\widehat{\mathbf{p}}^{\text{ML}}, \widehat{\theta}^{\text{ML}}\right\} = \quad & \underset{\{p_{\text{ctr.}},\Delta\mathbf{p},\theta\}}{\text{argmin}} \quad J(p_{\text{ctr.}}, \Delta\mathbf{p}, \theta), \quad (4) \\ & \text{s.t.} \qquad \sum_i \Delta p_i = 0. \end{aligned}$$

The previous reformulation allows us to prove the following lemma, which suggests how the ML estimator exploits the GPS information to solve for the absolute positioning of the formation centroid:

Lemma III.1. *Consider the negative* log*-likelihood cost function* (1)*. Then, the maximum likelihood solution* $\widehat{\mathbf{x}}^{\text{ML}}$ *which solves* (4) *is such that*

$$\widehat{p}_{\text{ctr.}}^{\text{ML}} = p_{\text{ctr.}}^{\text{GPS}}, \qquad (5)$$

where $\widehat{p}_{\text{ctr.}}^{\text{ML}} := \frac{1}{N}\sum_{i=1}^{N} \widehat{p}_i$ *and* $p_{\text{ctr.}}^{\text{GPS}} := \frac{1}{N}\sum_{i=1}^{N} p_i^{\text{GPS}}$.

Proof. The proof is reported in Appendix A. □

We can also state some limit behavior in a scenario where range, bearing and compass noises are very large or very small:

Lemma III.2. *For fixed GPS variance* σ_p *we have*

1) $\lim_{\max\{\sigma_\theta,\sigma_r,\sigma_\delta\}\to 0} \widehat{p}_i^{\text{ML}} = p_{\text{ctr.}}^{\text{GPS}} + \Delta p_i$,
2) $\lim_{\min\{\sigma_r,\sigma_\delta\}\to +\infty} \widehat{p}_i^{\text{ML}} = p_i^{\text{GPS}}$.

Proof. The proof is reported in Appendix B. □

Scenario 1) of Lemma III.2 states that in the case where $\max\{\sigma_\theta, \sigma_r, \sigma_\delta\} \to 0$, the shape of the formation is perfectly retrieved. In this case the only source of error between the estimated formation and the ground-truth is given by the error between GPS centroid and the true centroid. Scenario 2) states that if the relative measurements accuracies deteriorate, the ML estimator will "trust" the GPS measurements only. Unfortunately problem (2) is highly non linear and hard to solve. In particular, it is known that, if the angles are noise-free, the problem is linear [11]. Conversely, if the angles are not known, the problem presents many local minima [10], [22]. One possible way to tackle it, is using a standard gradient descent approach since the gradient vector of the log-likelihood function can be computed in closed form using (1). However, such approach heavily suffers of bad initialization. In fact, the presence of multiple local minima in the cost function (1) causes the algorithm to stop in the wrong minimizer.
In the following, we resort to a suitable approximation which let us reformulate the problem in a classical linear-least square framework.

C. An Approximated Linear Least-Squares Formulation

An approximated solution for the problem stated in (2), which exploits a suitable model linearization, is now presented. The idea is to move from the polar coordinate system to the equivalent Cartesian representation.
Indeed, assuming a perfect knowledge of range, bearing and compass, it is possible to express the *displacement* d_{ij} between agent i and j as

$$d_{ij} := p_i - p_j = r_{ij}\begin{bmatrix}\cos(\delta_{ij} + \theta_i) \\ \sin(\delta_{ij} + \theta_i)\end{bmatrix}. \qquad (6)$$

Since the measurements are affected by noise, it is necessary to map the noise of range, bearing and compass into the equivalent noise in Cartesian coordinates. Namely, given the noisy version of (6), that is

$$d_{ij} = p_i - p_j + n_{ij}, \qquad (7)$$

where n_{ij} is the noise in Cartesian coordinate, we want to find the expression for its covariance, $\mathbb{E}[n_{ij}n_{ij}^T] = \Sigma_{ij}$, in terms of the statistical description of range, bearing and compass measurements noises. After a first order expansion we obtain

$$\Sigma_{ij} = \begin{bmatrix}\sigma_x^2(i,j) & \sigma_{xy}(i,j) \\ \sigma_{yx}(i,j) & \sigma_y^2(i,j)\end{bmatrix}, \qquad (8)$$

where

$$\begin{aligned} \sigma_x^2(i,j) &= \sigma_r^2\cos^2(\delta_{ij}+\theta_i) + r_{ij}^2(\sigma_\delta^2+\sigma_\theta^2)\sin^2(\delta_{ij}+\theta_i), \\ \sigma_y^2(i,j) &= \sigma_r^2\sin^2(\delta_{ij}+\theta_i) + r_{ij}^2(\sigma_\delta^2+\sigma_\theta^2)\cos^2(\delta_{ij}+\theta_i), \\ \sigma_{xy}(i,j) &= \left(\sigma_r^2 - r_{ij}^2(\sigma_\delta^2+\sigma_\theta^2)\right)\sin(\delta_{ij}+\theta_i)\cos(\delta_{ij}+\theta_i). \end{aligned}$$

Remark III.2. Since the linear approximation introduced is based on a first order expansion, its validity holds under the assumption of sufficiently small measurement errors.

Remark III.3. Note that Σ_{ij} is a function of the true values of range, bearing and compass. Since it is not possible to have access to these data, in a real setup these quantities must be replaced by their corresponding measured values.

Once computed the displacements, it is possible to define the weighted residuals as

$$J_d = \frac{1}{2} \sum_{\{i,j\} \in \mathbf{E}} \| p_i - p_j - d_{ij} \|^2_{\Sigma_{ij}^{-1}}.$$

Thanks to this, it is possible to define an approximation of the negative log-likelihood in (1), which accounts for the GPS measurements and the displacements, as

$$J_{\text{LS}}(\mathbf{p}) = J_p + J_d. \tag{9}$$

The minimization problem becomes

$$\widehat{\mathbf{p}}^{\text{LS}} = \text{argmin}_{\mathbf{p}} \; J_{\text{LS}}(\mathbf{p}) \;, \tag{10}$$

which is a linear least-squares problem, thus convex, which can be solved in closed form. Specifically, assuming $\mathbf{G}$ connected, the optimal estimate is given by

$$\widehat{\mathbf{p}}^{\text{LS}} = (\Sigma_{\text{GPS}}^{-1} + A^T \Sigma^{-1} A)^{-1} (\Sigma_{\text{GPS}}^{-1} \mathbf{p}^{\mathbf{GPS}} + A^T \Sigma^{-1} \mathbf{d}), \tag{11}$$

where $\Sigma_{\text{GPS}} = \sigma_p^2 \mathbb{I}$, Σ is the matrix which accounts for all the Σ_{ij}, and $\mathbf{d}$ and $\mathbf{p}^{\text{GPS}}$ are the vectors obtained stacking together all the relative distances defined in (7) and the GPS absolute positions, respectively.

Remark III.4. Note that the LS estimates only the absolute positions $\mathbf{p}$ without providing any estimate of the absolute orientations. These are retrieved using the compass and exploited to project the noise in rectangular coordinates.

Remark III.5. Observe that, even if the linear least-squares problem returns an approximate solution for the problem of equation (2), since the problem of equation (10) is convex, its solution is unique.

For the LS estimator it is possible to show an optimal result similar to the one stated in Lemmas III.1 and III.2 for the ML estimator. We state the following:

Lemma III.3. *Consider the cost function* (9). *Then, the optimal solution* $\widehat{\mathbf{p}}^{\text{LS}}$ *which solves* (10) *is such that*

$$\widehat{p}_{\text{ctr.}}^{\text{LS}} = p_{\text{ctr.}}^{\text{GPS}}. \tag{12}$$

Moreover, for fixed GPS variance σ_p *we have*

$$\lim_{\max\{\sigma_\theta, \sigma_r, \sigma_\delta\} \to 0} \widehat{p}_i^{\text{LS}} = p_{\text{ctr.}}^{\text{GPS}} + \Delta p_i^{\text{LS}} \;,$$

$$\lim_{\min\{\sigma_r, \sigma_\delta\} \to +\infty} \widehat{p}_i^{\text{LS}} = p_i^{\text{GPS}} \;.$$

Proof. The result follows with arguments similar to those used in Lemma III.1 and III.2. □

Observe that, to compute $\widehat{\mathbf{p}}^{\text{LS}}$ as in equation (10), one needs all the measurements, their covariances and the topology of $\mathbf{G}$ to be available to a central computation unit. In the following section we present a solution which is amenable for a distributed and asynchronous implementation. We assume that a nodes i and j can communicate with each other only if $\{i,j\} \in \mathbf{E}$. Remarkably, the solution is robust to packet losses and delays in the communication channel.

IV. DISTRIBUTED AND ASYNCHRONOUS ALGORITHM

In this section we present a distributed and asynchronous solution for the minimization problem (10), which is robust to communication delays and packet losses. The implementation presented is inspired by [23], where is shown that this strategy is efficient both in terms of number of iterations and number of sent packets per communication round, compared to existing alternative strategies.

In the following:

1) by *distributed*, we mean that there is no central unit gathering all the measurements $\mathbf{p}^{\text{GPS}}$ and $\mathbf{d}$, having global knowledge of the graph $\mathbf{G}$ and computing $\widehat{\mathbf{p}}^{\text{LS}}$ directly; instead, each node has limited computational and memory resources, and can communicate only with its neighbors;
2) by *asynchronous*, we mean that there is no common reference time (generated, e.g., by a centralized clock source) which keeps all the updating/transmitting actions synchronized among all the nodes.

The algorithm we propose is based on a standard gradient descent strategy and employs an *asynchronous broadcast* communication protocol; specifically during each iteration of the algorithm there is only one node which transmits information to all its neighbors in the graph $\mathbf{G}$. Furthermore, the time between two consecutive iterations does not have to be constant. We refer to this algorithm as the *asynchronous gradient-based localization* algorithm (denoted hereafter as a-GL algorithm). For the sake of simplicity, from now on, the superscript LS in the single node estimates will be dropped. We assume that every node has access to its own measurements and the ones of its neighbors nodes, as well as the associated covariances. Additionally we assume that node i, $i \in \mathbf{V}$, stores in memory an estimate $\widehat{p}_{\text{i}}$ of p_i and, for $j \in \mathcal{N}_i$, an estimate $\widehat{p}_j^{(i)}$ of $\widehat{p}_j$.

The a-GL algorithm is shown in Algorithm 1. Let $t_0, t_1, t_2, \ldots$ be the time instants in which the iterations of the a-GL algorithm occur.

In Algorithm 1, $\alpha(i) = [\alpha_x(i) \;\; \alpha_y(i)]^T$ is a suitable scale factor for the gradient step. Through standard algebraic computations, one can see that:

$$\frac{\partial J_{\text{LS}}}{\partial p_i} = \frac{p_i - p_i^{\text{GPS}}}{\sigma_p^2} + \sum_{j \in \mathcal{N}_i} \Sigma_{ij}^{-1} (p_i - p_j - d_{ij}) \;.$$

Observe that in order to compute $\frac{\partial J_{\text{LS}}}{\partial p_i}$, node i requires information only from its neighbors. This makes the algorithm amenable for a distributed implementation. Since every node has available in memory a copy of the neighbors estimate, a

Algorithm 1: a-GL Algorihtm

Require: Node $i \in \mathbf{V}$ store in memory the measurements p_i^{GPS}, d_{ij}, $j \in \mathcal{N}_i$, the variances σ_p, N_{ij} and the neighbors estimates $\widehat{p}_j^{(i)}$, $j \in \mathcal{N}_i$.

1: **for** $t = t_0, t_1, t_2, \ldots$ **do**

\# Random node selection

2: Node $i \in \mathbf{V}$ wakes-up

\# Node i self update

3: $\widehat{p}_i \leftarrow \widehat{p}_i - \alpha(i) \odot \frac{\partial J_{\text{LS}}}{\partial p_i}$

\# Self-update broadcasting

4: $\widehat{p}_i$ broadcast to j, $j \in \mathcal{N}_i$

\# neighbors memory update

5: $\widehat{p}_i^{(j)} \leftarrow \widehat{p}_i, \forall j \in \mathcal{N}_i$

6: **end for**

natural way to evaluate the gradient is

$$\frac{\partial J_{\text{LS}}}{\partial p_i} = \frac{\widehat{p}_i(t) - p_i^{\text{GPS}}}{\sigma_p^2} + \sum_{j \in \mathcal{N}_i} \Sigma_{ij}^{-1} \left(\widehat{p}_i(t) - \widehat{p}_j^{(i)}(t) - d_{ij} \right) \,,$$

It is possible to show that J_{LS} does not increase if

$$0 < \alpha_x(i) \leq \left(\frac{1}{\sigma_p^2} + \sum_{j \in \mathcal{N}_i} \left(\gamma_x(i,j) + \gamma_x(j,i) \right) \right)^{-1} \,, \quad \text{(13a)}$$

$$0 < \alpha_y(i) \leq \left(\frac{1}{\sigma_p^2} + \sum_{j \in \mathcal{N}_i} \left(\gamma_y(i,j) + \gamma_y(j,i) \right) \right)^{-1} \,, \quad \text{(13b)}$$

where $\gamma_x(h,k)$ and $\gamma_y(h,k)$ represent the diagonal elements of Σ_{ij}^{-1}. In particular, if $\alpha(i)$ coincides with the RHS of (13a) then the minimum of J_{LS} is attained.
In the following we analyze the convergence properties and the robustness to packet losses and delays of the a-GL algorithm.

A. Convergence Analysis in Presence of Packet Losses and Communication Delays

In Algorithm 1 is presented a way to compute the linear least-squares solution of (10) in a distributed and asynchronous fashion. In this section we consider an even more realistic scenario: *presence of delays and packet losses in the communication channel*. Convergence of the a-GL algorithm to the optimal LS solution is proven, provided that the network is uniformly persistent communicating and the transmission delays and the frequencies of communication failures satisfy mild conditions which we formally describe next. We introduce the following definition.

Definition 1 (Uniformly persistent comm. network)**.** *A network of N nodes is said to be a* uniformly persistent communicating network *if there exists a positive integer number τ such that, for all $t \in \mathbb{N}$, each node perform lines 3 and 4 of the a-GL algorithm at least once within the iteration-interval* $[t, t+\tau)$.

Moreover, the following assumptions characterize the communication non-idealities.

Assumption IV.1 (Bounded packet losses)**.** There exists a positive integer L such that the number of consecutive communication failures between every pair of neighboring nodes in the graph $\mathbf{G}$ is less than L.

Assumption IV.2 (Bounded delays)**.** Assume node i broadcasts its estimate to its neighbors during iteration t, and, assume that, the communication link (i,j) does not fail. Then, there exists a positive integer D such that the information $\widehat{p}_i(t+1)$ is used by node j to perform its local update not later than iteration $t+D$.

Loosely speaking, Assumption IV.1 implies that there can be no more than L consecutive packet losses between any pair of nodes i, j belonging to the communication graph. Differently, Assumption IV.2 considers the scenario where the received packets are not used instantaneously, but are subject to some delay no greater than D iterations.
The following result characterizes the convergence properties of the a-GL algorithm in the scenario described by Definition 1 under Assumptions IV.1 and IV.2.

Proposition 1 (Proposition V.3 in [24])**.** *Consider a* uniformly persistent communicating network *of N nodes running the a-GL algorithm over a connected measurement graph* $\mathbf{G}$. *Let Assumptions IV.1 and IV.2 be satisfied. Assume the weights $\alpha(i)$ satisfy Equations* (13a)–(13b)*. Moreover, assume that $\widehat{p}_i$, $i \in \{1, \ldots, N\}$, $\widehat{p}_j^{(i)}$, $j \in \mathcal{N}_i$, be initialized to* $\mathbf{p}^{\text{GPS}}$*. Then the following facts hold true*

1) *the evolution $t \to \widehat{\mathbf{p}}(t)$ asymptotically converges to the optimal estimate $\widehat{\mathbf{p}}^{\text{LS}}$, i.e.,*
$$\lim_{t \to \infty} \widehat{\mathbf{p}}(t) = \widehat{\mathbf{p}}^{\text{LS}};$$
2) *the convergence is exponential, namely, there exists $C > 0$ and $0 \leq \rho < 1$ such that*
$$\|\widehat{\mathbf{p}}(t) - \widehat{\mathbf{p}}^{\text{LS}}\| \leq C \rho^t \|\widehat{\mathbf{p}}(0) - \widehat{\mathbf{p}}^{\text{LS}}\|. \quad (14)$$

Proof. The proof can be found in [24]. □

V. SIMULATIONS

In this section, we test the effectiveness of the proposed algorithm. We consider a group of robots:

- placed on a $2D$ lattice formation;
- regularly spread with an inter-node distance of 4 meters.

We assume each agent to be endowed with:

- a GPS sensor characterized, according to [18], [19], by $\sigma_p = 2$ [m];
- a compass sensor characterized by, according to [20], by $\sigma_\theta = 0.05$ [rad];
- a range and a bearing sensors with standard deviations σ_r and σ_δ, respectively. Acceptable values are $\sigma_r = 0.1$ [m] and $\sigma_\delta = 0.03$ [rad]. However, due to their variability, we test our algorithm in a sufficiently wide range of standard deviation values.

The reminder of the section is organized as follows:

1) in Section V-A, we briefly describe the performance measures used, later on, to test our algorithms;
2) in Section V-B, we analyze the steady state behavior of the a-GL algorithm with respect to the ground truth, for increasing number of nodes N and for different values of σ_r and σ_δ;
3) in Section V-C, we analyze the transient behavior (convergence analysis) of the a-GL algorithm in terms

of number of iterations with respect to the optimal configuration obtained from (11).

A. Performance Measures

For the steady state analysis of Section V-B, the estimated positions are compared with the ground truth in terms of Mean Squared Error (*MSE*). Specifically, by denoting the generic vector of positions estimate $\widehat{\mathbf{p}} = [\widehat{p}_1, \ldots, \widehat{p}_N]^T$ where $\widehat{p}_i = (\widehat{x}_i, \widehat{y}_i)$, the MSE of the positions is equal to

$$MSE(\widehat{\mathbf{p}}, \mathbf{p}) = \mathbb{E}\left[\|\widehat{\mathbf{p}} - \mathbf{p}\|^2\right] . \tag{15}$$

By defining the centroids of the estimated x and y coordinates as

$$\widehat{x}_{\text{ctr.}} := \frac{1}{N}\sum_{i=1}^{N} \widehat{x}_i \ , \qquad \widehat{y}_{\text{ctr.}} := \frac{1}{N}\sum_{i=1}^{N} \widehat{y}_i \ ,$$

and those of the true x and y coordinates as

$$x_{\text{ctr.}} := \frac{1}{N}\sum_{i=1}^{N} x_i \ , \qquad y_{\text{ctr.}} := \frac{1}{N}\sum_{i=1}^{N} y_i \ ,$$

the *MSE* can be rewritten as

$$MSE(\widehat{\mathbf{p}}, \mathbf{p}) = \mathbb{E}\Big[\sum_{i=1}^{N} \big((\widehat{x}_i - \widehat{x}_{\text{ctr.}}) - (x_i - x_{\text{ctr.}}) + (\widehat{x}_{\text{ctr.}} - x_{\text{ctr.}})\big)^2 + \big((\widehat{y}_i - \widehat{y}_{\text{ctr.}}) - (y_i - y_{\text{ctr.}}) + (\widehat{y}_{\text{ctr.}} - y_{\text{ctr.}})\big)^2\Big] .$$

It is convenient to define the displacements from the centroid and the difference between the centroids for the x coordinate as

$$\Delta x_i := x_i - x_{\text{ctr.}}, \qquad \Delta \widehat{x}_i := \widehat{x}_i - \widehat{x}_{\text{ctr.}}, \qquad \Delta x_{\text{ctr.}} := \widehat{x}_{\text{ctr.}} - x_{\text{ctr.}} \ ,$$

and similarly Δy_i, $\Delta \widehat{y}_i$ and Δy_C those for the y coordinate. We recall the fact that

$$\sum_{i=1}^{N} \Delta x_i = \sum_{i=1}^{N} \Delta y_i = \sum_{i=1}^{N} \Delta \widehat{x}_i = \sum_{i=1}^{N} \Delta \widehat{y}_i = 0 \ .$$

After some algebraic manipulations it is possible to write

$$MSE(\widehat{\mathbf{p}}, \mathbf{p}) = MSE_{\text{Ctr.}} + MSE_{\text{Rel.Disp.}} \ ,$$

where

$$MSE_{\text{Ctr.}} := \mathbb{E}\left[\Delta x_C^2 + \Delta y_C^2\right] \ , \tag{16a}$$

$$MSE_{\text{Rel.Disp.}} := \mathbb{E}\left[\sum_{i=1}^{N} (\Delta \widehat{x}_i - \Delta x_i)^2 + (\Delta \widehat{y}_i - \Delta y_i)^2\right], \tag{16b}$$

represent the *MSE* of the centroids and of the relative displacement from the centroid, respectively. Note that the

$$MSE_{\text{Ctr.}} = \frac{\sigma_p^2}{N} \ ,$$

so, it scales with the number of nodes and tends to zero as $N \to \infty$.

For the transient analysis of Section V-C, we compare the performance of the a-GL algorithm with the steady state estimate obtained with the LS centralized algorithm, i.e.,

$$\|\widehat{\mathbf{p}}(t) - \widehat{\mathbf{p}}^{\text{LS}}\|. \tag{17}$$

As shown in equation (14), the a-GL exponentially converges to the centralized solution.

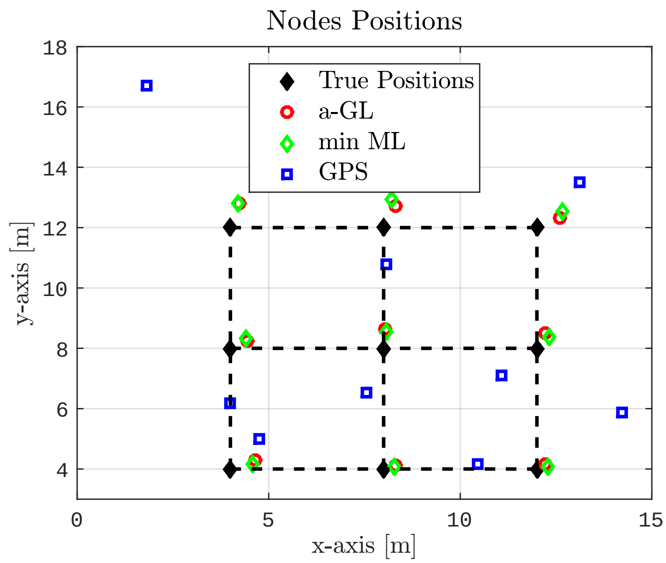

Fig. 1: Absolute positions for a formation of robots with $N = 9$, $\sigma_r = 0.1$ [m] and $\sigma_\delta = 0.03$ [rad]. The black dashed line highlights the shape of the real formation.

Remark V.1 (Numeric *MSE*)**.** Observe that the theoretic *MSE* cannot be exactly computed. In the following, we plot the numeric *MSE* computed via Monte Carlo simulations.

Remark V.2 (Dependence between σ_r and σ_δ)**.** In the following we test the proposed algorithm as a function of the relative measurements standard deviations, σ_r and σ_δ. We vary only the range standard deviation since the bearing measurements accuracy is assumed to depend on the range accuracy as $\sigma_\delta = atan2(\sigma_r, \frac{4}{3})$ which let us approximately draw samples in a ball centered in the true positions.

B. Steady State Analysis

In this section, we analyze the steady state behavior of the a-GL algorithm for increasing N and for different values of σ_r and σ_δ.

Figure 1 shows the absolute positions of the GPS measurements, the a-GL estimate and the minimizer of the log-likelihood, respectively. It can be seen how, thanks to the additional relative information, the estimates outperforms the GPS measurements.

Remark V.3. As outlined, the ML estimation problem is highly non-linear and characterized by many local minima. Then, the ML estimate has been computed by exhaustive search around the ground truth.

Figure 2 shows the behavior of the *MSE* of equation (15) for increasing N. Specifically, the *MSE* has been split into its components related to the centroid and the relative displacement, equations (16a)–(16b), respectively. It can be seen how the $MSE_{\text{Ctr.}}$ tends to zero for $N \to \infty$, while $MSE_{\text{Rel.Disp.}}$ remains almost constant and on the same order of magnitude of σ_r^2. From the plot, it can be understood that there are mainly two sources of error: one related to the absolute position reconstruction, which is obtained from the GPS information; one depending on the relative information. Thanks to accurate relative information, it is possible to reconstruct

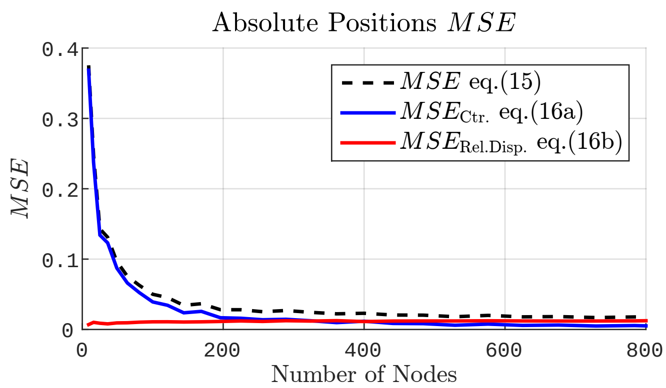

Fig. 2: Absolute positions *MSE*, computed via Monte Carlo simulations, as function of the number of nodes for $\sigma_r = 0.1$ [m] and $\sigma_\delta = 0.03$ [rad].

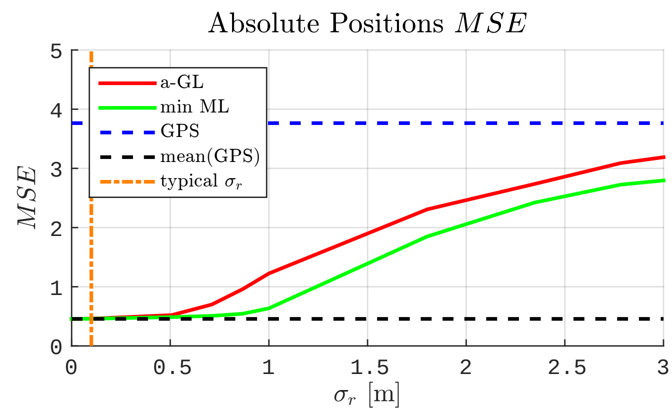

Fig. 3: Absolute positions *MSE*, computed via Monte Carlo simulations, as function of σ_r $\left(\sigma_\delta = atan2(\sigma_r, \frac{4}{3})\right)$ for $N = 9$. The dark orange vertical dashed-dotted line highlights the behavior corresponding to $\sigma_r = 0.1$ [m].

the shape of the formation with an error comparable to that of the relative measurements. The absolute formation position, which is recovered from the GPS, for small number of agents is the greater source of error, but improves with the number of robots as $\frac{1}{N}$. As already outlined, the proposed solution can be used seamlessly in scenario where not all the robots are equipped with GPS sensors. In this case, the absolute positions error scales with the number of agents equipped with a GPS module.

Figure 3 shows the absolute positions *MSE*, equation (15), for increasing values of σ_r. The plot shows the behavior of the a-GL algorithm (red line) compared with the behavior of the maximum likelihood estimator (green line). Moreover, some limit behaviors are plotted: the *MSE* of the GPS measurements (blue dashed line); the *MSE* of the mean of the GPS measurements (black dashed line). The limit behaviors, according to Lemma III.3, are due to the following facts:

- for increasing values of σ_r the relative sensors information becomes useless and the estimator will "trust" mainly the GPS measurements;
- for small values of σ_r the shape of the formation is "perfectly" known. So, the only source of error is due to the displacement of the GPS mean from the ground truth mean.

Figure 3 shows how the a-GL algorithm behaves similarly to the ML estimator for the whole range of σ_r. In addition to this, for values of σ_r within $0.1 \div 0.5$ [m], which characterize practical operating sensors range, the a-GL algorithm mimics almost perfectly the ML estimator.

C. Transient Analysis

In this section we analyze the transient behavior of the a-GL algorithm in presence of packet losses and communication delays. At each iteration, a node, randomly chosen, wakes up, updates its state and communicates its estimate to node $j \in \mathcal{N}_i$. We assume independent communication links between neighboring nodes, each of them characterized by a certain failure probability. Figure 4 plots, in logarithmic scale, the error in (17) between the a-GL estimated formation and the optimal LS solution, computed using (11). The different lines correspond to different percentages of packet losses. As expected, the higher is the losses the slower is the convergence. Note that, in a real set-up, different nodes could wake up and update their estimates at the same time. This could increase the possibility of communication collision but at the same time could speed up the convergence rate.

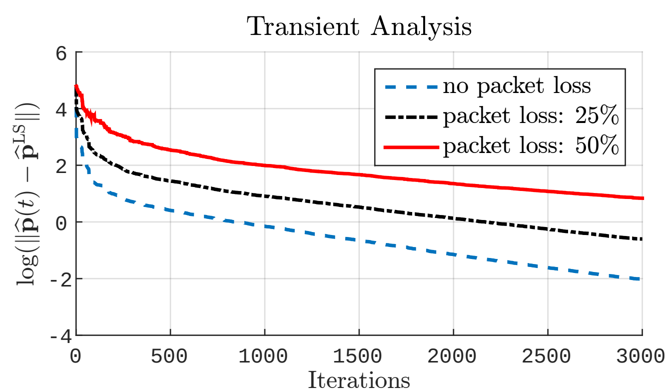

Fig. 4: Comparison between the a-GL solution and the optimal centralize LS solution, for different percentages of packet losses.

VI. CONCLUSIONS AND FUTURE WORK

In this work we presented an algorithm to solve the problem of absolute position reconstruction of a multi-robots formation. In particular, it is assumed each agent to be endowed with standard noisy GPS and compass modules and finer relative range and bearing sensors. Combining the absolute and relative information, we showed how the absolute global formation can be reconstructed. Specifically, a fast distributed and asynchronous Linear Least-Squares algorithm which solves an approximation of the Maximum Likelihood estimation problem is presented. Moreover, the algorithm is shown to be robust to delays and packet losses in the communication channel. Exhaustive numerical simulations show how, for sufficiently small relative errors, the approximated solution behaves like the ML estimator.

As future research directions, we will investigate the impact of the formation shape and of the communication graph on the relative formation reconstruction. Moreover, a solution which, filtering the absolute and relative angles measurements, could provide a better estimate of the robots absolute rotations will be analyzed.

APPENDIX

A. Proof of Lemma III.1

Observe that only the term J_p of the log-likelihood cost function depends on $p_{\text{ctr.}}$. Indeed, J_θ is not a function of p_i; while, both J_r and J_δ depend only on the difference between p_i and p_j which, thanks to the equation (3) reads as

$$p_i - p_j = p_{\text{ctr.}} + \Delta p_i - p_{\text{ctr.}} - \Delta p_j = \Delta p_i - \Delta p_j.$$

It is then possible to consider only the log-likelihood relative to the GPS measurements. Specifically, if we define $p_i^{\text{GPS}} = p_{\text{ctr.}}^{\text{GPS}} + \Delta p_i^{\text{GPS}}$, it is possible to write

$$\begin{aligned} 2\sigma_p^2 J_p &= \sum_{i=1}^{N} \|p_{\text{ctr.}} + \Delta p_i - (p_{\text{ctr.}}^{\text{GPS}} + \Delta p_i^{\text{GPS}})\|^2 \\ &= \sum_{i=1}^{N} \big(\|p_{\text{ctr.}} - p_{\text{ctr.}}^{\text{GPS}}\|^2 + \|\Delta p_i - \Delta p_i^{\text{GPS}}\|^2 + \\ &\quad +2(\Delta p_i - \Delta p_i^{\text{GPS}})^T (p_{\text{ctr.}} - p_{\text{ctr.}}^{\text{GPS}})\big) \\ &= N\|p_{\text{ctr.}} - p_{\text{ctr.}}^{\text{GPS}}\|^2 + \sum_{i=1}^{N} \|\Delta p_i - \Delta p_i^{\text{GPS}}\|^2, \end{aligned}$$

where we used the facts $\sum_i \Delta p_i = 0$ and $\sum_i \Delta p_i^{\text{GPS}} = 0$. To minimize the first term on the right hand side we must have

$$p_{\text{ctr.}} = p_{\text{ctr.}}^{\text{GPS}},$$

which proves the lemma.

B. Proof of Lemma III.2

In the first scenario $\max\{\sigma_\theta, \sigma_r, \sigma_\delta\} \to 0$. This implies that the distributions for compass, range and bearing measurements converge to delta distributions, implying that

$$r_{ij} \to \|p_i - p_j\|, \quad \theta_i^C \to \theta_i, \quad \delta_{ij} \to \theta_i + atan2(p_j - p_i).$$

From these expressions it easily follows that

$$\widehat{p}_j - \widehat{p}_i \to p_j - p_i = r_{ij} e^{j(\delta_{ij} - \theta_i^C)}, \quad \{j,i\} \in \mathbf{E} \ ,$$

i.e., the relative vectorial distances among the communicating nodes are perfectly known. Since the graph is connected, it is possible to compute the exact vectorial difference among any two agents in the network, and therefore also the exact distance of any agent from the true centroid since:

$$\Delta\widehat{p}_i = \widehat{p}_i - \frac{1}{N}\sum_j \widehat{p}_j = \frac{1}{N}\sum_j (\widehat{p}_i - \widehat{p}_j) \to \frac{1}{N}\sum_j (p_i - p_j) = \Delta p_i.$$

Since $\widehat{p}_i = \widehat{p}_{\text{ctr.}} + \Delta\widehat{p}_i$ and from Lemma III.1 we have $\widehat{p}_{\text{ctr.}} = p_{\text{ctr.}}^{\text{GPS}}$, then it follows the first part of the lemma.
In the second scenario when $\min\{\sigma_r, \sigma_\delta\} \to +\infty$ becomes arbitrary large, the probability distribution of range and bearing degenerate into an uniform distribution with infinite support. As so, the terms J_r and J_δ become negligible as compared to J_p and J_θ. Since the positions p_i do not appear in J_θ, it follows that $\widehat{p}_i$ results from the minimization of J_p, which gives $\widehat{p}_i = p_i^{\text{GPS}}$ and, therefore, the claim of the lemma.

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