Multi-agent perimeter patrolling subject to mobility constraints

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Abstract— **In this paper we study the problem of real-time optimal distributed partitioning for perimeter patrolling in the context of multi-camera networks for surveillance. The objective is to partition a given segment into non-overlapping sub-segments, each assigned to a different camera to patrol. Each camera has both physical mobility range and limited speed, and it must patrol its assigned sub-segment by sweeping it back and forth at maximum speed. Here we first review the solution for the centralized optimal partitioning. Then we propose two different distributed control strategies to determine the extremes of the optimal patrolling areas of each camera. Both these strategies require only local communication with the neighboring cameras but adopt different communication schemes, respectively, symmetric gossip and asynchronous asymmetric broadcast. The first scheme is shown to be provably convergent to the optimal solution. Some theoretical insights are provided also for the second scheme whose effectiveness is validated through numerical simulations.**

I. Introduction

The task of patrolling refers to the act of an agent that senses a different portion of the environment at a time, in order to detect events or anomalies: The possibility of controlling the information acquisition is central to the task and characterizes the patrolling agent both as a sensor and as an actuator.

The problem of patrolling has been and is extensively studied in the framework of mobile agents (see for example [5][9][6] and references within) where a set of robots or other autonomous vehicles moves in the environment in order to attain the optimal area coverage in a dynamic sense and to act coordinately according to the information they gather and share. In this case, the mobile agents are typically constrained by their motion dynamics, by their sensing capabilities, and by the communication protocols. Indeed, there may be need for the agents to exchange information only when in close contact [8], or for the whole system to trade off with finite capacity issue in wireless links.

In [2], the case of smart camera patrolling is studied as a particular case of the multiagent network. In this work, the problem of patrolling is cast into a pursuit-evasion paradigm: Two players are considered, a rational patroller and a rational intruder and game-theory techniques are applied in order to design an optimal patrolling strategy for the patroller, in terms of Pan-Tilt-Zoom (PTZ) commands.

A smart camera network is introduced in our previous work [1], where the considered problem is that of patrolling executed by a network of PTZ cameras in a typical outdoor videosurveillance setup. Differently from [2], the particular focus here is on the coordination and communication among cameras, which is studied in the context of distributed algorithms. In this scenario, the case of perimeter patrolling is considered and each camera is located in a fixed position with limited visibility of the scene and limited motion capability: Since the patrolling task corresponds to the action of visually monitoring the environment, each camera needs to coordinate its motion with the its neighbors in order to ensure an optimal coverage policy of the whole monitored area.

The paper is organized as follows. The case of perimeter patrolling is studied reducing the domain of interest to a one dimensional domain, as formally detailed in Section II and in Section III. In Section IV and in Section V the main contributions are presented, related respectively to the symmetric gossip and the asynchronous asymmetric broadcast algorithms. We conclude in Section VI with some numerical examples.

II. Perimeter patrolling

In this section we review the problem of patrolling a one-dimensional environment of finite length with a finite number of cameras and its optimal solution as described in [1].

Specifically let $\mathcal{L} = [0, L]$, $L > 0$, denote the segment to be monitored and let N be the cardinality of the camera set. The cameras are labeled 1 through N and, for the sake of simplicity, we assume that (a) the cameras are 1-d.o.f., meaning that the field of view (f.o.v.) of each camera is allowed to change due to pan movements only; (b) the cameras have fixed coverage range, meaning that during pan movements the camera coverage range is not altered by the view perspective; (c) cameras have point f.o.v..

In this context we also introduce the following definitions. The *patrolling range* D_i is the total potential area that i-th camera can patrol due to the scenario topology, the agent configuration and their physical constraints. More formally $D_i = \left[\underline{d}_i, \overline{d}_i\right] \subset \mathcal{L}$, $\underline{d}_i < \overline{d}_i$, where $\underline{d}_i$, $\overline{d}_i$ are the left and the right extreme of the interval D_i, respectively. The *max speed* $\overline{v}_i \in \mathbb{R}^+$ is the maximum speed of i-th camera during pan movements, i.e. $|v_i(t)| \leq \overline{v}_i$. The *patrolling area* A_i: denotes the area that is actively patrolled by the i-th camera where, clearly, it must hold $A_i \subseteq D_i$, $\forall\, i \in \{1, \dots, N\}$ where $A_i = [\ell_i, r_i]$, being ℓ_i, r_i, respectively, the left and right extreme of A_i. In our analysis, we assume that the

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coverage ranges D_i, $i \in \{1, \dots, N\}$, satisfies the following *interlacing* physical coverage constraint,

$$\underline{d}_i \leq \underline{d}_{i+1} \leq \overline{d}_i \leq \overline{d}_{i+1}, \quad i = 1, \dots, N-1. \tag{1}$$

Moreover we impose that $\underline{d}_1 = 0$ and $\overline{d}_N = L$. These conditions guarantee that the area can be fully patrolled, i.e. $\cup_{i=1}^{N} D_i = \mathcal{L}$.

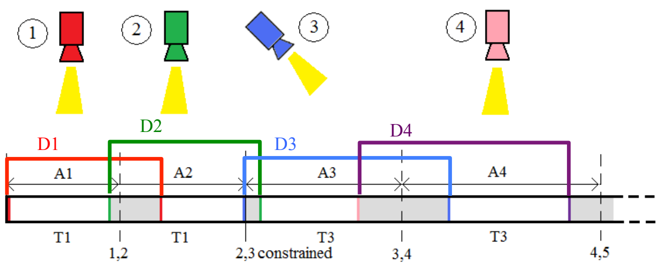

Fig. 1. Perimeter patrolled by a camera set. For the first four cameras, the physical coverages $\{D_i\}$ with some overlapping sections are shown, together with the optimal partition domains $\{A_i\}$.

In order to properly define the patrolling problem we need to introduce an appropriate cost function J and state an optimality criterium. The *camera position* $z_i(t) : \mathbb{R}^+ \rightarrow D_i$ is the the position of the f.o.v. of the i-th camera as a function of the time variable t. The authors in [1] propose a functional J whose rationale is as follows: at each time instant t and position $x \in \mathcal{L}$, J is equal to 0 if location x is currently seen by any camera ($\exists\, i\ s.t.\ z_i(t) = x$), else it takes a positive real value which is a monotonic function of the *patrolling time lag* T_{lag} defined as the maximum (w.r.t. $x \in \mathcal{L}$) elapsed time between two visits of the same location, therefore the minimization problem for J corresponds to the computation of the smallest time lag T_{lag}, constrained to the system dynamics

$$\dot{z}_i(t) = v_i(t) \quad s.t. \left\{ \begin{array}{l} |v_i(t)| \leq \overline{v}_i \\ z_i(t) \in D_i \end{array} \right. . \tag{2}$$

We now consider the problem of optimally patrol, i.e., of minimizing the patrolling time lag, a certain area A_i by the i-th camera. The next proposition states which is the smallest achievable patrolling time lag and the corresponding camera motion.

Proposition II.1 *Let $A_i = [\ell_i, r_i]$ the patrolling area assigned to the i-th camera. Then the minimum patrolling time lag T_i^* within this area is given by*

$$T_i^* = T_{lag}^*(A_i) = \frac{2|A_i|}{\overline{v}_i} = \frac{2(r_i - \ell_i)}{\overline{v}_i} \tag{3}$$

and is achieved by assuming that camera i moves at its maximum speed $\bar{v}_i$ sweeping back and forth A_i with a periodical motion of period T_i^*.

The previous proposition states that once a patrolling area A_i is assigned to a camera, the optimal strategy is to sweep back and forth this area at maximum speed. As a consequence the problem of globally minimizing the patrolling time lag reduces to the problem of design the patrolling area A_i, which is a partitioning problem. More formally, we are interested in solving the following optimization problem

$$\mathcal{P}_1 : \quad \begin{array}{lll} \min_{A_1,\dots,A_N} & \max_i \{T_{lag}^*(A_i)\} & (4) \\ s.t. & A_i \subseteq D_i \quad i = 1, \dots, N & (5) \\ & \cup_{i=1}^{N} A_i = \mathcal{L} & (6) \end{array}$$

where Eqn. (4) represents the objective which is the minimization of the largest patrolling time leg among all patrolling areas A_i, Eqn. (5) represents the physical constrains arising from the limited patrolling range of the cameras, and Eqn. (6) represents the requirement that all points in $\mathcal{L}$ are eventually visited. The next proposition shows that the previous problem can be re-casted as a linear program (LP), i.e., it can efficiently solved using standard optimization software.

Proposition II.2 *Consider the problem $\mathcal{P}_1 := (T_{\mathcal{P}_1}^*, X_{\mathcal{P}_1})$ defined by Equations (4)-(6) where $T_{\mathcal{P}_1}^*$ is the minimum and $X_{\mathcal{P}_1} = \{\{A_i\}_{i=1}^N \mid \max_i\{T_{lag}^*(A_i)\} = T_{\mathcal{P}_1}^*\}$ is the set of minimizers. Then the optimization problem $\mathcal{P}_1$ is equivalent to the following LP problem:*

$$\mathcal{P}_2 : \quad \begin{array}{lll} \min_{\tau, \{r_i\}_{i=1}^{N-1}, \{\ell_i\}_{i=2}^{N}} 2\tau & & (7) \\ s.t.\ \frac{r_i - \ell_i}{\overline{v}_i} \leq \tau & i = 1, \dots, N & (8) \\ \underline{d}_i \leq \ell_i \leq \overline{d}_i, \quad \underline{d}_i \leq r_i \leq \overline{d}_i & i = 1, \dots, N & (9) \\ r_i \geq \ell_{i+1} & i = 1, \dots, N-1 & (10) \end{array}$$

where $\underline{d}_1 = \ell_1 = 0$ and $\overline{d}_N = r_N = L$

Proof: The equivalence between the two optimization problem $\mathcal{P}_1$ and $\mathcal{P}_2$ is obtained by adding a slack variable τ to transform the max function into a linear function subject to linear inequalities, which give rise to Eqn. (7) and (8). Eqn. (9) and Eqn. (10) are simply the reformulation of Eqn. (5) and Eqn. (6) in terms of the representation of the areas A_i through their extremes (ℓ_i, r_i). ■

The previous proposition provides a centralized solution to the patrolling problem but cannot be easily computed in a distributed fashion. Although distributed algorithm exists for the solution of LP problems [7], these involve the solution of the all problem at each node, i.e., each camera would compute the optimal patrolling areas A_i also for all other cameras, which is not necessary since each camera needs to compute only its own optimal patrolling area A_i. Moreover, the previous optimization problem might have multiple minimizers. The next proposition states that it is possible to reformulate a new optimization problem $\mathcal{P}_3$ which has the properties that its minimizer is unique and it is also a minimizer for the original problem $\mathcal{P}_1$.

Proposition II.3 *Consider the following optimization problem*

$$(\mathcal{P}_3) \quad \begin{array}{lll} \min_{\{r_i\}_{i=1}^{N-1}, \{\ell_i\}_{i=2}^{N}} \sum_{i=1}^{N} \frac{1}{\overline{v}_i}(r_i - \ell_i)^2 & & (11) \\ \underline{d}_i \leq \ell_i \leq \overline{d}_i, \quad \underline{d}_i \leq r_i \leq \overline{d}_i & i = 1, \dots, N & (12) \\ r_i \geq \ell_{i+1} & i = 1, \dots, N-1 & (13) \end{array}$$

where $\underline{d}_1 = \ell_1 = 0$ and $\overline{d}_N = r_N = L$. Let $X_{\mathcal{P}_3}$ the corresponding set of minimizers. Then $X_{\mathcal{P}_3} = \{(A_1^*, \dots, A_N^*)\}$ is a singleton and $X_{\mathcal{P}_3} \subseteq X_{\mathcal{P}_1}$.

Proof: For the sake of clarity, the proof is divided into several steps:

(a) Without loss of generality we identify a partition set $\{A_1, \dots, A_N\}$ with its set of free extremes $\xi := (r_1, \ell_2, r_2, \dots, \ell_N) \in \mathbb{R}^{2N-2}$ since by definition $\ell_1 = 0$ and $r_N = L$. Consider the function

$$f(\xi) = \frac{1}{\overline{v}_1} r_1^2 + \sum_{i=2}^{N-1} \frac{1}{\overline{v}_i}(r_i - \ell_i)^2 + \frac{1}{\overline{v}_N}(L - \ell_N)^2$$

which is the objective function of Eqn. (11). This function is quadratic and positive definite which implies that it is also strictly convex. The constraints set $\mathcal{C}$ defined by Eqn. (12) and Eqn. (13) is convex, compact and non-empty, therefore the minimum of the function $f(\cdot)$ restricted to the set $\mathcal{C}$ exists and it is unique. We refer to this minimum as $\xi^* \equiv (A_1^*, \dots, A_N^*)$. This proves the first part of the proposition.

(b) We now show that $|A_i^* \cap A_{i+1}^*| = 0$ for $i = 1, \dots, N-1$, i.e.

$$r_i^* = \ell_{i+1}^* \tag{14}$$

Assume by contradiction that this is not true, i.e. there is i such that $r_i^* > \ell_{i+1}^*$. From Eqn. (1) it follows that $\ell_{i+1}^* \geq \underline{d}_{i+1} \geq \underline{d}_i$ and $r_i^* \leq \overline{d}_i \leq \overline{d}_{i+1}$. Let us consider the new partition $\xi' = \xi^*$ for all elements except for r_i' which is set to $r_i' = \ell_{i+1}^*$. This choice is feasible, i.e. $\xi' \in \mathcal{C}$, since $r_i' \geq \ell_{i+1}' = \ell_{i+1}^* \geq \underline{d}_i$. This leads to $f(\xi^*) - f(\xi') = \frac{1}{\overline{v}_i}((r_i^* - \ell_i^*)^2 - (\ell_{i+1}^* - \ell_i^*)^2) > 0$, which contradicts the assumption that ξ^* is the global minimum of f over $\mathcal{C}$.

(c) Let us define $T_{\mathcal{P}_3}^* := \max_i\{T_{lag}^*(A_i^*)\} = \max_i \left\{\frac{|A_i^*|}{\overline{v}_i}\right\}$. Let j be the index of most left region for which the maximum period is achieved, i.e.

$$j := \operatorname{argmin}\{i \,|\, T_{lag}^*(A_i^*) = T_{\mathcal{P}_3}^*\}$$

and similarly let

$$h = \operatorname{argmin}\{h \,|\, T_{h+1}^* < T_{\mathcal{P}_3}^* \text{ or } h = N\}$$

be the index of the last contiguous region with the maximum period. Clearly, this implies that that $T_{lag}^*(A_j^*) = T_{lag}^*(A_{j+1}^*) = \cdots = T_{lag}^*(A_h^*) = T_{\mathcal{P}_3}^*$. We now show that

$$\ell_j^* = \overline{d}_{j-1}, \quad r_h^* = \underline{d}_{h+1} \tag{15}$$

where we adopted the little abuse of notation $\overline{d}_0 := \underline{d}_1 = 0$ and $\underline{d}_{N+1} := \overline{d}_N = L$. If $j > 1$, then by definition $T_{leg}^*(A_{j-1}^*) < T_{leg}^*(A_j^*)$ which implies that $\frac{r_j^* - r_{j-1}^*}{\overline{v}_j} - \frac{r_{j-1}^* - \ell_{j-1}^*}{\overline{v}_{j-1}} = \epsilon > 0$ where we used Eqn (14). We now show that we must have $\ell_j^* = r_{j-1}^* = \overline{d}_{j-1}$. Assume by contradiction that $\ell_j^* = r_{j-1}^* < \overline{d}_{j-1}$. Let us define the positive scalar δ as follows

$$\delta := \min\left(\frac{\overline{v}_{j-1}\overline{v}_j}{\overline{v}_{j-1} + \overline{v}_j}\epsilon \,,\, \overline{d}_{j-1} - r_{j-1}^*\right)$$

Consider the new feasible partition ξ' where $\xi' = \xi^*$ for all elements except for $r_{j-1}' = \ell_j' = r_{j-1}^* + \delta$. Clearly $\xi' \in \mathcal{C}$. This leads to

$$f(\xi') - f(\xi^*) = \frac{(r_j^* - r_{j-1}^* - \delta)^2}{\overline{v}_j} + \frac{(r_{j-1}^* - \ell_{j-1}^* + \delta)^2}{\overline{v}_{j-1}} - \frac{(r_j^* - r_{j-1}^*)^2}{\overline{v}_j} - \\ - \frac{(r_{j-1}^* - \ell_{j-1}^*)^2}{\overline{v}_{j-1}} = \delta\left(-2\frac{r_j^* - r_{j-1}^*}{\overline{v}_j} + 2\frac{r_{j-1}^* - \ell_{j-1}^*}{\overline{v}_{j-1}} + \frac{\delta}{\overline{v}_{j-1}} + \frac{\delta}{\overline{v}_{j-1}}\right) \\ \leq \delta(-2\epsilon + \epsilon) = -\delta\epsilon < 0$$

which contradicts the hypothesis that ξ^* is the unique minimizer of f over $\mathcal{C}$. The same line of reasoning can be employed to show that $r_h^* = \ell_{h+1}^* = \underline{d}_{h+1}$ if $h < N$.

(d) We now want to show that $\xi^* \in X_{\mathcal{P}_1}$, i.e. $T_{\mathcal{P}_3}^* = T_{\mathcal{P}_1}^*$. Assume by contradiction that $T_{\mathcal{P}_3}^* > T_{\mathcal{P}_1}^*$. From Eqns. (14)-(15) it follow that $\cup_{i=1}^N A_i^* = [\overline{d}_{j-1}, \underline{d}_{h+1}]$ and $|A_i^* \cap A_{i+1}^*| = 0$, therefore $\sum_{i=j}^h |A_i^*| = \underline{d}_{h+1} - \overline{d}_{j-1}$. Let us now consider any optimal partition for the original problem $\mathcal{P}_1$, i.e. $(A_1, \dots, A_N) \in X_{\mathcal{P}_1}$, then $T_{lag}^*(A_i) \leq T_{\mathcal{P}_1}^*, \forall i$, which implies that $|A_i| \leq \frac{1}{2}\overline{v}_i T_{\mathcal{P}_1}^* < \frac{1}{2}\overline{v}_i T_{\mathcal{P}_3}^* = |A_i^*|$. Moreover, we must have $\cup_{i=j}^h A_i \supseteq [\overline{d}_{j-1}, \underline{d}_{h+1}]$ since $\ell_j \leq r_{j-1} \leq \overline{d}_{j-1}$ and $r_h \geq \ell_{h+1} \geq \underline{d}_{h+1}$ from Eqns. (9)-(10). This implies that $\sum_{i=j}^h |A_i| \geq \underline{d}_{h+1} - \overline{d}_{j-1}$ which contradicts the observation that $\sum_{i=j}^h |A_i| \leq \sum_{i=j}^h \frac{1}{2}\overline{v}_i T_{\mathcal{P}_1}^* < \sum_{i=j}^h \frac{1}{2}\overline{v}_i T_{\mathcal{P}_3}^* = \sum_{i=j}^h |A_i^*| = \underline{d}_{h+1} - \overline{d}_{j-1}$. Therefore $T_{\mathcal{P}_3}^* = T_{\mathcal{P}_1}^*$, which concludes the proof. ■

From the proof of the previous proposition, it follows the following corollary that shows that for unlimited patrolling range the optimal partitioning corresponds to assign to each camera an area whole length is proportional to its pan speed, i.e. faster cameras patrol longer perimeter segments. If in addition the camera speeds are all equal, then the optimal partitioning is the equal partitioning, as one would intuitively expect.

Corollary II.4 *Let us consider the optimization problem $\mathcal{P}_1$ and $\mathcal{P}_1$ without patrolling range constraints given by Eqn. (5) and Eqn. (12), i.e. $\underline{d}_i = 0$ and $\overline{d}_i = L$ for all i. Then*

$$|A_i| = \frac{\overline{v}_i}{\sum_{i=1}^N \overline{v}_i} L, \quad \forall i$$

and in particular $|A_i| = \frac{L}{N}$ if $\overline{v}_i = \overline{v}, \forall i$.

The benefits of the optimization problem $\mathcal{P}_3$ as compared to the optimization problem $\mathcal{P}_1$ is twofold. The first benefit is that under specific communication strategies, namely gossip communication, it can be solved with distributed algorithms which are scalable and parallelizable, as shown in the next sections. The second benefit is that uniqueness of the minimizer in $\mathcal{P}_3$ guarantees the practical convergence of iterative numerical algorithms to a unique point, which otherwise as in the case of $\mathcal{P}_1$ might oscillate within the set $X_{\mathcal{P}_1}$.

III. Distributed optimal partitioning: problem formulation

In this section we consider the partitioning problems $\mathcal{P}_1$ and $\mathcal{P}_3$ within an "iterative" and "distributed" scenario. Specifically, we assume each camera is initialized at time $t = 0$ with a partition $A_i(0)$ that, in general, does not coincide with the optimal solution. Each camera is allowed to iteratively update A_i using only the local information coming from the neighboring cameras. The goal is to provide strategies that lead the cameras to asymptotically reach the optimal steady-state configuration for patrolling extremes.

In the next section we formally describe the setup we consider and the problem we aim to solve.

We assume that at time $t = 0$ each camera is initialized with a *dominance interval* $A_i(0)$. More precisely, for $i \in \{1, \ldots, N\}$ let $A_i(0) = [\ell_i(0), r_i(0)]$ where $\ell_i(0)$ and $r_i(0)$ are respectively the left extreme and the right extreme of $A_i(0)$. We assume that the set $\{A_1(0), \ldots, A_N(0)\}$ satisfies two constraints. Firstly, we assume a *physical constraint*, that is,

$$A_i(0) \subseteq D_i, \qquad i \in \{1, \ldots, N\}. \tag{16}$$

Secondly, a *interlacing constraint* is posed, that is,

$$\ell_i(0) \leq \ell_{i+1}(0) \leq r_i(0) \leq r_{i+1}(0). \tag{17}$$

Moreover we impose the following *boundary conditions*

$$\ell_i(0) = \underline{d}_1, \qquad r_N(0) = \overline{d}_N. \tag{18}$$

Observe that from (17) and (18) it follows that the set $\{A_1(0), \ldots, A_N(0)\}$ satisfies also the following *covering constraint*

$$\bigcup_{i \in \{1,\ldots,N\}} A_i(0) = \mathcal{L}.$$

The goal is to design iterative partitioning algorithms that allow the cameras to update their dominance intervals using only information coming from neighboring cameras and such that

(i) the physical constraints, the interlacing constraints and the boundary conditions, introduced in (16), (17) and (18), respectively, are satisfied at each iteration; and
(ii) the set of dominance intervals converge to the optimal partition.

It is worth clarifying that for neighboring cameras we mean that camera i, $i \in \{2, \ldots, N-1\}$ exchange information with camera $i-1$ and camera $i+1$. If $i = 1$ (resp. $i = N$) the only neighbor of camera 1 (resp. N) is camera 2 (resp. $N-1$).

In next sections we consider two different communication protocols adopted by the cameras to exchange information with each other. More precisely, in Section IV we consider a symmetric gossip-type communication protocol; specifically, at each iteration of the partitioning algorithm only a pair of neighboring cameras communicate with each other while the other cameras do not transmit or receive any information. In this context we introduce the *symmetric-gossip partitioning algorithm*.

In Section V we relax the communication-symmetry required in the previous Section and we consider an asymmetric gossip-type communication protocol. While in the symmetric gossip the active communication link is bidirectional, that is, if camera i transmits to camera $i+1$, then at the same time camera $i+1$ transmits to camera i, in the asymmetric gossip only one direction is active, that is, either camera i transmits to camera $i+1$ or camera $i+1$ transmits to camera i. Accordingly we introduce the *asymmetric broadcast partitioning algorithm*.

IV. The symmetric-gossip partitioning algorithm

In this section we introduce the *symmetric-gossip partitioning algorithm* (denoted as *s-PA* hereafter). The algorithm is formally described as follows.

Processor states: For each $i \in \{1, \ldots, N\}$, camera i keeps in memory the extremes defining its *dominance interval*, i.e., ℓ_i and r_i. Moreover, we assume also that each camera knows the maximum patrolling-speed of its neighboring cameras;

Initialization: For $i \in \{1, \ldots, N\}$ values $\ell_i(0), r_i(0)$ are given as part of the problem. We assume that the initial conditions satisfy the *interlacing* and *physical* constraints and the *boundary conditions*.

Transmission iteration: For $t \in \mathbb{N}$, during the t-th iteration of the *s-PA*, only a pair of neighboring cameras, say i and $i+1$, communicate with each other; the communicating link is bidirectional, namely, camera i sends to camera $i+1$ the values of its extremes $\ell_i(t)$ and $r_i(t)$ and, camera $i+1$ sends to camera i the values of its extremes $\ell_{i+1}(t)$ and $r_{i+1}(t)$;

Extremes' iteration: For $h \notin \{i, i+1\}$, camera h left unchanged its extremes, that is, $\ell_h(t+1) = \ell_h(t)$ and $r_h(t+1) = r_h(t)$.

Camera i and camera $i+1$, based on the received information, update r_i and ℓ_{i+1} performing the following two actions. First they compute the point p^* according to the following *neighbors' equal traveling time* criterion

"*the time required to camera i to travel at the speed v_i from p^* to $\ell_i(t)$ is equal to the time required by the camera $i+1$ to travel at speed v_{i+1} from p^* to $r_{i+1}(t)$*"; Formally p^* satisfies the condition

$$\frac{p^* - \ell_i(t)}{\bar{v}_i} = \frac{r_{i+1}(t) - p^*}{\bar{v}_{i+1}} \tag{19}$$

which yields

$$p^* = \frac{\ell_i(t)\,\bar{v}_{i+1} + r_{i+1}(t)\,\bar{v}_i}{\bar{v}_i + \bar{v}_{i+1}}. \tag{20}$$

Second both cameras i and $i+1$ check if the intervals $[\ell_i(t), p^*]$ and $[p^*, r_{i+1}(t)]$ satisfy the physical constraints and they update r_i and ℓ_{i+1} by setting

$$r_i(t+1) = \ell_{i+1}(t+1) = p^*$$

if $p^* \in \left[\underline{d}_{i+1}, \overline{d}_i\right]$, that is, if $[\ell_i(t), p^*] \subseteq D_i$ and $[p^*, r_{i+1}(t)] \subseteq D_{i+1}$, otherwise they set

$$r_i(t+1) = \ell_{i+1}(t+1) = \begin{cases} \overline{d}_i & \text{if} \quad p^* > \overline{d}_i \\ \underline{d}_{i+1} & \text{if} \quad p^* < \underline{d}_{i+1} \end{cases}$$

Moreover for all $t \in \mathbb{N}$ we have that

$$\ell_1(t) = 0 \qquad \text{and} \qquad r_N(t) = L.$$

We characterizes now the convergence properties of the *s-PA*. We provide conditions ensuring both deterministic and probabilistic convergence. We start with the deterministic convergence.

Theorem IV.1 *Consider the* s-PA. *Let* $\{A_i(0)\}_{i=1}^N$ *be the initial set of dominance intervals which is assumed to satisfy the* boundary *conditions, the* interlacing *and* physical *constraints. Moreover assume that there exists a positive integer number* τ *such that, for all* $t \in \mathbb{N}$*, any pair of neighboring cameras* $(i, i+1)$*,* $i \in \{1, \ldots, N-1\}$*, communicates with each other at least once within the interval* $[t, t+\tau)$*. Then the evolution* $t \to \{A_i(t)\}$ *generated by the* s-PA *algorithm satisfies:*

(i) *the* boundary *conditions, the* interlacing *and* physical *constraints are verified for all* $t \in \mathbb{N}$*; and*
(ii) *the set* $\{A_i(t)\}_{i=1}^N$ *converges asymptotically to the optimal solution* $\{A_i^*\}_{i=1}^N$ *of Problem* $\mathcal{P}_3$ *and, in turn, to one optimal solution of Problem* $\mathcal{P}_1$.

Proof: The fact that the *boundary* conditions, the *interlacing* and *physical* constraints are verified for all $t \in \mathbb{N}$, easily follows from the description of the *s-PA*. We prove now that the sequence $\{A_i(t)\}_{i=1}^N$ converges asymptotically to an optimal solution of Problem $\mathcal{P}_3$. We start by observing that after τ iterations of the *s-PA* algorithm, we have that the right extreme of camera i is equal to the left extreme of camera $i+1$, i.e, $r_i(t) = \ell_{i+1}(t)$ for $t \geq \tau$. For $t \geq \tau$, let us introduce the auxiliary variables $x_1(t), \ldots, x_{N-1}(t)$ where $x_i(t) = r_i(t) = \ell_{i+1}(t)$. Define $x(t) = [x_1(t), \ldots, x_{N-1}(t)]$ and notice that, according to the physical constraints, $\underline{d}_{i+1} \leq x_i(t) \leq \overline{d}_i$. Let

$$W = \prod_{i=1}^{N-1} \left[\underline{d}_{i+1}, \overline{d}_i\right].$$

Since W is given by the Cartesian product of $N-1$ closed intervals, it follows that W is compact. Now, for $i \in \{1, \ldots, N-1\}$ let $T_i : W \to W$ be the map describing the **Extremes' iteration** of the *s-PA* in case the communicating pair of cameras is $(i, i+1)$. Observe that, for $i \in \{1, \ldots, N-1\}$, the map T_i is continuous with the respect to the standard Euclidean metric.

The goal is to apply Theorem VII.1 reported in Appendix. To this end, for $x = [x_1, \ldots, x_{N-1}] \in W$, let us introduce the function $U : W \to \mathbb{R}$ such that

$$U(x) = \frac{1}{\bar{v}_1} + \sum_{i=2}^{N-1} \frac{1}{\bar{v}_i} (x_i - x_{i-1})^2 + \frac{1}{\bar{v}_N}(L - x_{N-1})^2.$$

Observe that, for $i \in \{1, \ldots, N-1\}$, U is a convex parabola on the variable x_i having as vertex the point $\frac{\bar{v}_{i+1}x_{i-1}+\bar{v}_i x_{i+1}}{\bar{v}_i+\bar{v}_{i+1}}$ which, according to the definition of the variables $\{x_i\}_{i=1}^{N-1}$, coincides with the point p^* defined in (20).

Assume $x(t+1) = T_i(x(t))$ for some $i \in \{1, \ldots, N-1\}$ and observe that, according to the *s-PA*, either $x_i(t+1) \in [x_i(t), p^*]$ or $x_i(t+1) \in [p^*, x_i(t)]$. Hence, if $x_i(t+1) \neq x_i(t)$ then $U(x(t+1)) - U(x(t)) < 0$.

We are now in position of applying Theorem VII.1 and to conclude that $x(t)$ converges to the set

$$F_1 \cap \cdots \cap F_m,$$

where $F_i = \{x \in W \mid T_i(x) = x\}$ is the set of fixed points of T_i. Clearly $F_1 \cap \cdots \cap F_m$ is a singleton which coincides with the optimum of problem $\mathcal{P}_3$.

■

We provide now conditions ensuring probabilistic convergence.

Theorem IV.2 *Consider the* s-PA *algorithm. Let* $\{A_i(0)\}_{i=1}^N$ *be the initial set of dominance intervals which is assumed to satisfy the* boundary *conditions, the* interlacing *and* physical *constraints. Moreover assume that there exists a real number* $\bar{p}$*,* $0 < \bar{p} < 1$*, such that, for all* $t \in \mathbb{N}$ *and for all* $i \in \{1, \ldots, N-1\}$

$$\mathbb{P}\left[(i, i+1) \text{ communicates at iteration } t\right] \geq \bar{p}. \qquad (21)$$

Then the evolution $t \to \{A_i(t)\}$ *generated by the* s-PA *algorithm satisfies:*

(i) *the* boundary *conditions, the* interlacing *and* physical *constraints are verified for all* $t \in \mathbb{N}$*; and*
(ii) *the set* $\{A_i(t)\}_{i=1}^N$ *converges almost surely to the optimal solution* $\{A_i^*\}_{i=1}^N$ *of Problem* $\mathcal{P}_3$ *and, in turn, to one optimal solution of Problem* $\mathcal{P}_1$.

Proof:
The proof is based on the application of Theorem VII.2 reported in the Appendix and it is similar to the one of Theorem IV.1. ■

Remark IV.3 It is worth remarking that the *s-PA* was already introduced in [4]. However in [4] the authors provided the convergence of the *s-PA* only in the simplified scenario without physical constraints or, equivalently, when $D_1 = D_2 = \ldots = D_N = \mathcal{L}$.

V. The Asymmetric Broadcast Partitioning Algorithm

The aim of this section is to reduce the requirements of the *s-PA*, in terms of symmetric and reliable communications. To do so, we next introduce the *asymmetric broadcast partitioning algorithm* (denoted as *a-PA* hereafter). This algorithm is based on a asymmetric broadcast communication protocol. Differently from the *s-PA*, at each iteration of the *a-PA* there is only one camera transmitting information to its neighbors.

With the respect of the *s-PA*, the **Transmission iteration** and the **Extremes' update** are modified as follows

Transmission iteration: For $t \in \mathbb{N}$, during the t-th iteration of the algorithm there is only a camera, say i, which transmits information to its neighbors; precisely camera i sends to camera $i-1$ and camera $i+1$ the values of the extremes of its dominance region $A_i(t)$, i.e., $\ell_i(t), r_i(t)$.

Extremes' iteration: For $h \neq i-1, i+1$, camera h left unchanged its extremes, i.e., $\ell_h(t+1) = \ell_h(t)$ and $r_h(t+1) = r_h(t)$. Instead, based on the received information, camera $i-1$ and camera $i+1$ updates only the extremes "closer" to camera i. Specifically $\ell_{i-1}(t+1) = \ell_{i-1}(t)$ and $r_{i+1}(t+1) = r_{i+1}(t)$ while r_{i-1} and ℓ_{i+1} are updated as we next describe.

Similarly to (20), camera $i-1$ computes the point p^* such that

$$p^* = \frac{r_i(t)v_{i-1} + \ell_{i-1}(t)v_i}{v_{i-1}+v_i}$$

and updates r_{i-1} according to

$$r_{i-1}(t+1) = p^* \tag{22}$$

if the interlacing and physical constraints are satisfied, namely if $p^* \geq \ell_i(t)$ and $p^* \leq \overline{d}_{i-1}$, otherwise it sets

$$r_{i-1}(t+1) = \begin{cases} \overline{d}_{i-1} & \text{if} \quad p^* > \overline{d}_{i-1} \\ \ell_i(t) & \text{if} \quad p^* < \ell_i(t) \end{cases} \tag{23}$$

Analogously camera $i+1$ computes the point p^* such that

$$p^* = \frac{r_{i+1}(t)v_i + \ell_i(t)v_{i+1}}{v_i + v_{i+1}}$$

and it sets

$$\ell_{i+1}(t+1) = p^*,$$

if $p^* \leq r_i(t)$ and $p^* \geq \underline{d}_{i+1}$, otherwise

$$\ell_{i+1}(t+1) = \begin{cases} r_i(t) & \text{if} \quad p^* > r_i(t) \\ \underline{d}_{i+1} & \text{if} \quad p^* < \underline{d}_{i+1} \end{cases} \tag{24}$$

Clearly, if the transmitting camera is camera 1 (respectively camera N), then only camera 2 (respectively camera $N-1$) performs the extremes' updating action.

The following Proposition provides a desired characterization of the sequence $\{A_i^*(t)\}_{t\in\mathbb{N}}$.

Proposition V.1 *Consider the* a-PA. *Let* $\{A_i(0)\}_{i=1}^N$ *be the initial set of dominance intervals, which is assumed to satisfy the* interlacing *and* physical *constraints and the* boundary *conditions. Then, the evolution* $t \to \{A_i(t)\}_{i=1}^N$ *generated by the* a-PA *satisfies the property that the* interlacing *and* physical *constraint, the* boundary *conditions and, in turn, the* covering *conditions, are verified for all* $t \in \mathbb{N}$.

Unfortunately we were not able so far to theoretically characterize the convergence properties of the *a-PA*. However we run a number of simulations for different initial conditions, for different values of the camera velocities and for different physical constraints under the assumption of *Uniformly persistent transmissions*: namely, there exists an integer number τ such that, for all $t \in \mathbb{N}$, each camera performs a data transmission within the interval $[t, t+\tau]$. In all the cases the *a-PA* converged to an optimal partition of Problem $\mathcal{P}_1$.

VI. Numerical examples

In this section we provide two examples illustrating the effectiveness of *a-PA*.

Example VI.1 We consider a set of $N = 6$ cameras with the goal of patrolling the interval $\mathcal{L} = [0, 60]$. We assume that all the velocities $\bar{v}_i$, $i \in \{1, \ldots, N\}$, take the same value v and that at each iteration of the of the *a-PA*, a transmitting camera is randomly chosen in $\{1, \ldots, N\}$, with the constrain that, for all $t \in \mathbb{N}$, each camera is selected at least once within the time window $[t, t+\tau]$ with $\tau = 50$. Moreover for simplicity we assume that no physical constraints are present, or equivalently that $D_i = \mathcal{L}$, for all $i \in \{1, \ldots, N\}$.

In Figure 2 we plot the behavior of ℓ_i, r_i, $i \in \{1, \ldots, N\}$. The simulation shows how ℓ_{i+1} and r_i converge to the same value.

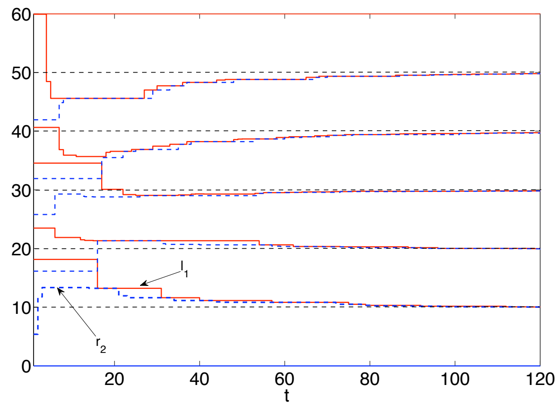

Fig. 2. Behavior of the extremes when *a-PA* is applied with $N = 6$ cameras.

Example VI.2 We consider a set of 50 cameras with the goal of patrolling the interval $\mathcal{L} = [0, 200]$. We assume that the velocities are all equal to the same value v, i.e., $\bar{v}_i = v$, for all $i \in \{1, \ldots, N\}$. We assume that at each iteration of the *a-PA*, a transmitting camera is randomly chosen in $\{1, \ldots, N\}$, with the constrain that, for all $t \in \mathbb{N}$, each camera is selected at least once within the time window $[t, t+\tau]$ with $\tau = 200$.

To evaluate the performance of *a-PA* we consider the following functional cost

$$J(t) = \frac{1}{N}\sum_{i=1}^{50} \left(\ell_i(t) - \ell_i^*\right)^2 + \left(r_i(t) - \ell_{i+1}^*\right)^2$$

where $\{[\ell_i^*, r_i^*]\}$ represents the optimal solution of Problem $\mathcal{P}_3$.

The obtained result is plotted in Figure 3. Observe that J goes to 0 as t increases showing the effectiveness of the *a-PA*.

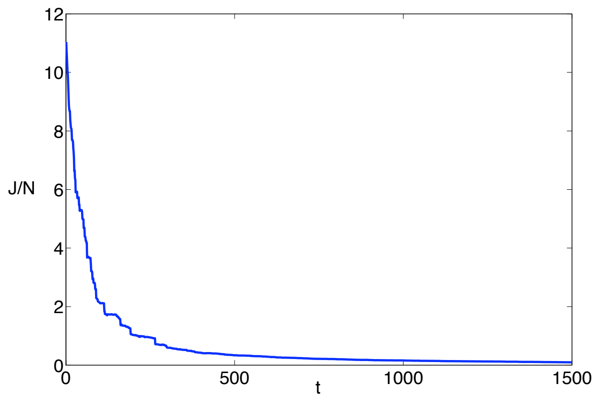

Fig. 3. Simulation of the a-PA with $N = 50$ cameras.

VII. Conclusions

In this paper, we studied the problem of real-time optimal perimeter partitioning taking into accounts physical, covering and mobility constraints. We proposed a symmetric gossip-type and asymmetric broadcast-type adaptive algorithms and we studied their properties analytically and numerically. Several future directions are possible. One of the most important is to provide analytical proof of convergence of the asymmetric broadcast-type algorithm.

Appendix

IN this appendix we review two convergence results for set-valued algorithms. In doing so, we follow the treatment in [3].

Given a set X, a set-valued map $T : X \rightrightarrows X$ is a map which associates to an element $x \in X$ a subset $Z \subset X$. A set-valued map is non-empty if $T(x) \neq \emptyset$ for all $x \in X$. A set $W \subset X$ is *strongly positively invariant* for T if $T(w) \subset W$ for all $w \in W$. Given a non-empty set-valued map T, an evolution of the dynamical system associated to T is a sequence $\{x_n\}_{n\in\mathbb{Z}_{\geq 0}} \subset X$ with the property that $x_{n+1} \in T(x_n)$ for all $n \in \mathbb{Z}_{\geq 0}$.

Theorem VII.1 *Let (X, d) be a metric space. Given a collection of maps $T_1, \dots, T_m : X \to X$, define the set-valued map $T : X \rightrightarrows X$ by $T(x) = \{T_1(x), \dots, T_m(x)\}$ and let $\{x_n\}_{n\in\mathbb{Z}_{\geq 0}}$ be an evolution of T. Assume that:*

(i) *there exists a compact set $W \subseteq X$ that is strongly positively invariant for T;*
(ii) *there exists a function $U : W \to \mathbb{R}$ such that $U(w') < U(w)$, for all $w \in W$ and $w' \in T(w) \setminus \{w\}$;*
(iii) *the maps T_i, for $i \in \{1, \dots, m\}$, and U are continuous on W; and*
(iv) *for all $i \in \{1, \dots, m\}$, there exists an increasing sequence of times $\{n_k \mid k \in \mathbb{Z}_{\geq 0}\}$ such that $x_{n_k+1} = T_i(x_{n_k})$ and $(n_{k+1} - n_k)$ is bounded.*

If $x_0 \in W$, there exists $c \in \mathbb{R}$ such that the evolution $\{x_n\}_{n\in\mathbb{Z}_{\geq 0}}$ approaches the set

$$(F_1 \cap \dots \cap F_m) \cap U^{-1}(c),$$

where $F_i = \{w \in W \mid T_i(w) = w\}$ is the set of fixed points of T_i in W, $i \in \{1, \dots, m\}$.

Next, we provide a probabilistic version of the previous theorem.

Theorem VII.2 *Let (X, d) be a metric space. Given a collection of maps $T_1, \dots, T_m : X \to X$, define the set-valued map $T : X \rightrightarrows X$ by $T(x) = \{T_1(x), \dots, T_m(x)\}$. Given a stochastic process $\sigma : \mathbb{Z}_{\geq 0} \to \{1, \dots, m\}$, consider an evolution $\{x_n\}_{n\in\mathbb{Z}_{\geq 0}}$ of T satisfying*

$$x_{n+1} = T_{\sigma(n)}(x_n).$$

Assume that:

(i) *there exists a compact set $W \subseteq X$ that is strongly positively invariant for T;*
(ii) *there exists a function $U : W \to \mathbb{R}$ such that $U(w') < U(w)$, for all $w \in W$ and $w' \in T(w) \setminus \{w\}$;*
(iii) *the maps T_i, for $i \in \{1, \dots, m\}$, and U are continuous on W; and*
(iv) *there exists $p \in]0, 1[$ and $k \in \mathbb{N}$ such that, for all $i \in \{1, \dots, m\}$ and $n \in \mathbb{Z}_{\geq 0}$, there exists $h \in \{1, \dots, k\}$ such that*

$$\mathbb{P}\big[\sigma(n + h) = i \mid \sigma(n), \dots, \sigma(1)\big] \geq p.$$

If $x_0 \in W$, then there exists $c \in \mathbb{R}$ such that almost surely the evolution $\{x_n\}_{n\in\mathbb{Z}_{\geq 0}}$ approaches the set

$$(F_1 \cap \dots \cap F_m) \cap U^{-1}(c),$$

where $F_i = \{w \in W \mid T_i(w) = w\}$ is the set of fixed points of T_i in W, $i \in \{1, \dots, m\}$.

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