

Algorithms for turbulence compensation in large adaptive optics systems

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Abstract—The ever growing demand of more resolution for ground telescopes makes of fundamental importance the use of computationally efficient algorithms. In this paper we consider some efficient algorithms for the adaptive optics system of large telescopes. The main peculiarities of the considered procedures are to be effective and scalable to telescopes of whatever size. In particular, we propose a decoupled representation of the turbulent phase in which each of the subsystems models the temporal dynamic of a turbulent mode (e.g. the evolution of a Zernike mode if using the Zernike bases to represent the turbulence). The model matrices are identified using recently developed subspace methods. Then, using it in a Kalman-based approach, it provides good performances for the closed-loop system. Furthermore, we analyze and compare other possible approaches, such as PI controllers and AR predictive models. Since computational efficiency plays a very important role in this framework, we evaluate the obtained results both for the absolute performances and for the computational efforts necessary to obtain them. The proposed Kalman-based model ensures a good tradeoff between complexity and performances. Anyway, when the system allows to use some more resources, it can be worth to consider the use of high order AR models.

I. INTRODUCTION

Adaptive optics (AO) systems are of fundamental importance to improve the real resolution of large telescopes. It is well known that the theoretical resolution of a lens is limited by the diffraction effect: For this reason the size of next generations of ground telescopes is progressively increasing. However, local changes of the atmospheric refraction index affect the phase of light wavefronts arriving on the telescope aperture, thus significantly reducing the real resolution of the telescope. Then, the aim of the AO system is that of compensating the turbulence effect allowing the telescope resolution be again diffraction limited.

An AO system is usually formed by a proper wavefront sensor, a set of deformable mirrors (DM) and a control unit which should compute proper inputs for the DM to adapt their shape to the current value of the phase distorted by the turbulence. The set of such distorted phases over the telescope aperture is also called the turbulent phase.

Here we consider a modal approach, that is the turbulent phase is projected on a set of spatial bases, both to reduce the signal dimension and the influence of noise. Then the projection of the phase on each this bases corresponds to a mode of the system.

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Two of the main issues in controlling the DM are that the measurements can be quite noisy and, due to the time necessary for image formation on the sensor, there is a two sampling period delay in the control action [9][8][10]. Hence the AO system performance largely depends on the ability of the control to adapt itself to the temporal evolution of the turbulent phase. Consequently, several control algorithms proposed in literature include a prediction step to reduce the effect of the delay. However, the computation of the proper correction is still a challenging task because the system usually works at high frequencies, therefore, the control action has to be sufficiently simple to allow for fast computation.

Specifically, in this paper we consider the case of algorithms for controlling the DM in particularly large telescopes: In fact, to make an algorithm scalable (that is to make it usable for whatever size of telescopes), it has to be computationally efficient, i.e. linear (or almost linear) in the number of actuators and sensor measurements (or in the number of modes). To make this possible, one cannot take into account of the relations between different modes: Thus the temporal cross-correlations are discarded. This results in a decoupled model of the system, where each of the modes is considered independently on the others.

Recently, several state-space models have been considered ([9][8][10],[12],[1]), with the aim of using a Kalman predictor both to compensate the delay effect and to reduce the noise influence. Motivated by these works, we propose a new state-space model of the turbulence, which is both computationally cheap and quite effective. The system is decomposed in a set of decoupled subsystems, where each of them models the temporal dynamic of a mode. The parameters of each of the subsystems are computed using a subspace identification algorithm, specifically the PBSID [5], [6]. Then, the identified model is used in a Kalman approach to predict the temporal evolution of the turbulence.

Furthermore, in Section IV and in our simulations, we consider also other possible control strategies, namely an approximation of the zonal control [11], a PI controller, and a prediction based control which uses AR models. Even if at the cost of a greater computational complexity and memory requirement, in our simulations high order AR models ensure the best performances.

Then, the choice between using the proposed Kalman-based approach or high order AR models depends on the allowed computational complexity and memory resources.

Finally, it may be worth to say that the proposed state-space model may be also included in other control strategies.

II. TURBULENT PHASE CHARACTERIZATION

The turbulent phase is assumed to be zero-mean stationary and spatially homogeneous. Let \mathbf{u} and \mathbf{v} be two unit vectors indicating two orthogonal spatial directions, and let $\phi(u, v, t)$ be the value of the turbulent phase on the point (u, v) at time t on the telescope aperture plane, where u and v are the coordinates of the point along \mathbf{u} and \mathbf{v} . Then, the covariance between two values of the turbulence, $\phi(u, v, t)$ and $\phi(u', v', t)$, depends only on the distance, r , between the two points: $C_\phi(r) = \mathbf{E}[\phi(u, v, t)\phi(u', v', t)], \forall(u, v, u', v')$, such that $r = \sqrt{(u - u')^2 + (v - v')^2}$.

According to the Von Karman theory, the shape of the spatial covariance function, $C_\phi(\cdot)$, is completely characterized by the values of two physical parameters, r_0 , the Fried parameter, and L_0 , the outer scale ([11], [7]).

In particular, roughly speaking, the Fried parameter approximately corresponds to the diameter of a circular area over which the variance of the phase distortion (without any AO correction) is 1 rad^2 [11]. Notice that this, 1 rad^2 , is commonly accepted as the maximum tolerable distortion to have usable astronomical observations.

From a temporal point of view, the turbulence is generally modeled as a (linear) superposition of a finite number l of independent layers, moving at different altitudes, with different energies and velocities. A commonly agreed assumption considers that each layer translates in front of the telescope pupil with constant velocity v_i (Taylor approximation [11]).

III. AO SYSTEM DESCRIPTION

To reduce the phase distortion due to the presence of the atmosphere, telescopes are provided with a set of deformable mirrors, which act like feedbacks on the turbulent phase arriving on the telescope aperture (see Fig. 1). Then the AO system is formed by the wavefront phase sensor, the deformable mirrors and their control unit. Its goal is that of properly controlling the deformable mirrors to compensate the turbulence effect.

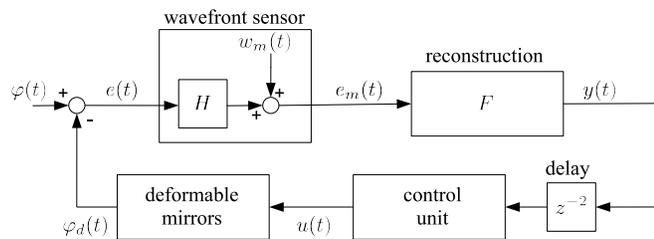


Fig. 1. Scheme of the adaptive optics system.

As shown in Fig. 1, the incoming turbulent phase $\varphi(t)$ is corrected by the current shape of the deformable mirrors, $\varphi_d(t)$. Then, a wavefront sensor provides a noisy measurement $e_m(t)$ related to the value of the residual phase $e(t)$. $e_m(t)$ is used to reconstruct the current uncorrected turbulent phase (typically projected on a set of spatial bases) $y(t)$. Finally the control unit takes $y(t)$ as input and computes a proper control for the deformable mirrors. Mostly due to

the time for image formation on the sensor, there is a 2-step delay before the control unit ([9], [8]), which has to compute the control $u(t)$ exploiting measurements only up to time $t - 2$.

By assumption the wavefront sensor provides measurements of the phase slopes (along both the spatial directions \mathbf{u} and \mathbf{v}) between the points on a grid \mathbb{L} over the telescope aperture. This, for instance, is the case of the most used wavefront sensor, that is the Shack-Hartmann device.

For small values of $e(t)$ (i.e. as far as the AO system is working properly), the wavefront sensor transfer function can be assumed to be static and linear, $e_m(t) = He(t) + w_m(t)$, where w_m is a zero-mean white noise, due to the measurement process. Let us call Q_m the covariance of w_m . For simplicity of notation, in the following we will assume that $Q_m = \sigma_m^2 I$, where σ_m^2 is the measurement noise variance.

Notice that, from an optical point of view, the phase translations over the entire telescope aperture can be neglected, hence the AO system does not take into account the projection of the turbulent phase on the vector $u_0 = [1 \ 1 \ \dots \ 1]^T$. Thus, we consider $\varphi(t)$ as the vector containing the values of the turbulent phase at time t neglecting its projection on u_0 . This is also in agreement with the assumptions taken on the wavefront sensor: Indeed, measurements of the slopes do not allow the reconstruction of the phase mean.

In this paper, we consider a modal approach, that is we represent the turbulent phase $\varphi(t)$ as projected on a set of spatial bases $U = [u_1 \ \dots \ u_n]$: $\varphi(t) = Ux(t) + e_r(t)$, where $e_r(t) = \varphi(t) - Ux(t)$ is the representation error and $x(t) = U^\dagger \varphi(t)$ (U^\dagger is the pseudo-inverse of U). In particular, we consider the bases provided by Principal Component Analysis (PCA) and we neglect the coefficients associated to low energy bases [3]. This can be thought both as a dimensionality reduction step ($n < |\mathbb{L}|$), and as a denoising procedure (the discarded coefficients are usually those more affected by noise).

Notice that, even if models in Section IV are presented assuming to use a PCA representation, in fact similar considerations can be repeated with minor changes for every other choice of the spatial bases. Analogously, we expect the results in Section V would not change very much using other bases.

The reconstruction procedure is performed by premultiplying $F = (HU)^\dagger$ to the measurement vector $e_m(t)$ (this is called a vector-matrix-multiply (VMM) reconstructor): $y(t) = Fe_m(t)$. Let $\varphi_d(t) = Ux_d(t) + e_d(t)$, where $x_d(t) = U^\dagger \varphi_d(t)$, then

$$y(t) = (x(t) - x_d(t)) + w(t), \quad (1)$$

where $w(t)$ is zero-mean and its covariance is $R \approx FF^T \sigma_m^2$. If not differently specified, hereafter we assume $R = FF^T \sigma_m^2$.

It is a common assumption ([11][9]) to take also the deformable mirrors transfer function as linear and static, i.e. $\varphi_d(t) = Du(t)$.

Then, the aim of the control unit is that of computing a proper $u(t)$, such that to minimize the energy of $\varphi(t) - \varphi_d(t)$,

that is to minimize $CE(t) = \text{Trace}(\text{var}(\varphi(t) - \varphi_d(t))) / m$, where $m = |\mathbb{L}|$ and $CE(t)$ is the coherent energy at time t (e.g. the variance of the residual phase).

In literature, deformable mirrors are usually characterized by the so called *interaction matrix* \bar{D} , which, using our notation, can be written as $\bar{D} = HD$. In practice, this corresponds to directly considering the relation between the voltages used as input for the deformable mirrors and the corresponding measured slopes.

Then, the simplest kind of control u used in AO system is that provided by the multiplication of measured current residuals $e_m(t)$ for the pseudo-inverse of the interaction matrix \bar{D}^\dagger : $u(t+2) = u(t+1) + \bar{D}^\dagger e_m(t)$. In this case the temporal delay is discarded (assuming $\varphi(t+2) \approx \varphi(t)$) and the aim is that of correcting the measured phase residual: $e_m(t) - (HD)\bar{D}^\dagger e_m(t) \approx 0$. For comparison with the other methods, in this paper we consider a simple approach similar to that described above: We compute $u(t+2)$ to minimize $y(t) - FHDu(t+2)$, then $u(t+2) = u(t) + (FHD)^\dagger y(t)$, which, after some simple matrix manipulations and exploiting the considered assumptions becomes $u(t+2) = u(t) + y(t)$. Hereafter this will be called zonal control (even if in fact it is only an approximation of the zonal control).

Finally, to give some intuition on the causes of error for the considered AO system, we distinguish three sources of error as follows: Consider the residual phase in a zonal control approach (recall that $D = U$),

$$\begin{aligned} \varphi(t) - \varphi_d(t) &= Ux(t) + e_r(t) - Du(t) \\ &= Ux(t) + e_r(t) - D(u(t-2) + y(t-2)) \\ &= Ux(t) + e_r(t) - U(x_d(t-2) + x(t-2) \\ &\quad - x_d(t-2) + w(t-2)) \\ &= Ux(t) + e_r(t) - Ux(t-2) + Uw(t-2) \\ &= U(x(t) - x(t-2)) + Uw(t-2) + e_r(t) . \end{aligned}$$

Since w is white it immediately follows that,

$$\text{var}(\varphi(t) - \varphi_d(t)) \approx U \text{var}(x(t) - x(t-2)) U^T + URU^T + \Sigma_{e_r},$$

where Σ_{e_r} is the covariance matrix of e_r . Then $CE(t)$ can be approximated by the sum of three terms $\epsilon_d(t) = \text{Trace}(\text{var}(x(t) - x(t-2)) U^T) / m$, $\epsilon_n(t) = \text{Trace}(URU^T) / m$, $\epsilon_r = \text{Trace}(\Sigma_{e_r}) / m$.

Since we are interested to the asymptotic behavior of the system, the time index t can be depicted from the equations. As already anticipated, the above error terms correspond to three different causes of error: $CE \approx \epsilon_d + \epsilon_n + \epsilon_r$. First, ϵ_r corresponds to the representation error, i.e. the error due to the dimensionality reduction step. ϵ_n is the component due to the presence of measurement noise, and ϵ_d is due to the temporal delay between the measurement instant and the application of the corresponding control.

We say that an AO system is *calibrated* when the above three terms are (approximatively) equal. Since this condition determines a good tradeoff between the requirements for the different parts of the system and the overall performances, it

is usually considered the preferable working condition whenever it is feasible. In a modal approach the representation error cannot be reduced (when the number of considered bases has already been chosen), thus the aim of the control approaches presented in the following section is to determine u to reduce the effect of the delay and of the measurement noise. We refer the reader to [11], [8], [1] for more detailed descriptions of AO systems.

IV. MODAL CONTROL FOR LARGE AO SYSTEMS

Due to the constant demand for more resolution, the lens size of next generations of ground telescopes is progressively growing. On the other side, the AO system often works at high frequencies, and the development of more effective photon sensors will probably make them even higher. Therefore, the control action has to be sufficiently simple to allow for fast computation, although measurement noise and the presence of the 2-step delay in the system have to be properly handled to reduce their effect.

It has already been shown ([2], [1]) that the coefficients in x are typically not temporally uncorrelated. Furthermore, especially when the sampling frequency is not so high (e.g. the system dynamic is richer), this temporal correlation can be exploited to improve the AO system performances. However, at high frequencies and for large systems (i.e. $n \approx 10^3$ or even larger) there might not be enough time to make possible the use of a model taking into account of cross-correlations between the spatial modes. Furthermore, such a model may also be particularly sensitive to changes in the operational conditions: e.g. turbulence characteristics may vary over time. Then either the model is updated frequently or its performances would probably progressively decrease. Instead, models not considering cross-temporal correlations are usually simpler to be updated and much less sensitive for instance to changes in the wind direction.

Thus, control approaches characterized by models with uncorrelated coefficients seem to be well suited for large telescopes. In particular, the goal is that of providing models which ensure good performances while requiring a linear (or almost linear) computational complexity with the size of the system (i.e. n). Since the complexity of the already presented ‘‘zonal control’’ is linear with respect to n , this will be considered for comparison as the simplest model satisfying the complexity requirements. Other possible choices for the control action will be presented in the following.

A. PI controller

A possible choice for the control is that of choosing to use a PI. Let a subscript index i indicate the i th coefficient of the corresponding vector, e.g. $y_i(t)$ denotes the i th coefficient in $y(t)$. Then, the considered PI controller is:

$$u_i(t) = u_i(t-1) + \left(k_P + \frac{k_I}{2}\right) y_i(t) + \left(-k_P + \frac{k_I}{2}\right) y_i(t-1),$$

for $i = 1, 2, \dots, n$. Notice that it has been used the following realization of a discrete integrator: $u_{i,I}(t) = \frac{k_I(1+z^{-1})}{2(1-z^{-1})} y_i(t)$. In the above equation k_P and k_I do not depend on i . While

this is of course a particularly simple case, it is also quite widely diffused. To ensure the stability of the closed-loop system, the values of k_P and k_I are allowed to be only in an interval between 0 and 1: The choice of the current values for k_P and k_I is taken looking for those which minimize the residual phase distortion in a set of learning samples.

B. Bayesian estimation and AR models

Undoubtedly, most of the difficulties in controlling large AO systems are due to the necessity of reducing the delay effect while using a computationally cheap approach.

If at time t we would know the value of $x(t)$, it would be quite simple to simply choose $u(t) = (FHD)^\dagger x(t)$. However, since $x(t)$ is unknown, we cannot directly apply the above control law, but, we can still use it after substituting $x(t)$ with its estimate $\hat{x}(t|t-2)$ obtained by using the available measurements up to time $t-2$:

$$u(t) = (FHD)^\dagger \hat{x}(t|t-2). \quad (2)$$

Then, the idea of prediction based methods, considered in this and in the following subsection, is that of providing a proper estimate $\hat{x}(t|t-2)$.

Exploiting temporal cross-correlations between the coefficients, $\{x_i\}$, is practically forbidden for computational complexity reasons (at least if not limited to small groups of coefficients, e.g. a block decoupled system). Thus, we consider only decoupled models for the coefficients, i.e. only past values of x_i can affect the value of $\hat{x}_i(t|t-2)$.

In a Bayesian framework we assume to have an a priori knowledge of the physical characteristics of the system, that is we assume to have estimations of the turbulent layer characteristics (velocities, energies, \hat{r}_0 , \hat{L}_0) and about the measurement noise, i.e. \hat{R} . Thus, we can compute estimations $\hat{C}_\phi(r)$ of $C_\phi(r)$, $\forall r$, and consequently also the estimate $\hat{\Sigma}_{x_i}(h)$ of $\Sigma_{x_i}(h) = \mathbf{E}x_i(t)x_i(t-h)$, $\forall h$.

Then, the Bayesian estimate of $x_i(t)$ given the previous available turbulent phase reconstruction (i.e. $\xi_i(t-2) \doteq u_i(t-2) + y_i(t-2) \approx x_i(t-2)$) can be written as follows: $\hat{x}_i(t|t-2) = \hat{\Sigma}_{x_i}(2)(\hat{\Sigma}_{x_i}(0) + \hat{R}(i,i))^{-1}\xi_i(t-2)$.

When the temporal dynamic of the coefficient x_i is complex (not well approximated by a first order model), it may be advantageous to use more past samples to get better estimations. Furthermore, if we do not have a priori information about the system, it may be of particular interest to estimates such model directly from the data. This is similar to the estimation of the coefficients for an AR model.

Let $\xi_i(1), \dots, \xi_i(T_i)$ be a set of learning samples collected running the AO system with a zonal control and summing the current deformable mirror value to the measured residual turbulent phase, i.e. $\xi_i(t) = u_i(t) + y_i(t)$. Consider the following (delayed) AR model, $\xi_i(t) = \sum_{h=2}^{\bar{h}} a_{ih}\xi_i(t-h) + \varepsilon_i(t)$, then the coefficients $\{a_{ih}\}$ are computed minimizing the L_2 norm of ε_i on the learning set.

The control law for this kind of model becomes,

$$u_i(t) = (FHD)_i^\dagger \left(\sum_{h=2}^{\bar{h}} a_{ih}\xi_i(t-h) \right),$$

where $(FHD)_i^\dagger$ indicates the i th row of $(FHD)^\dagger$.

Hereafter we will refer to $AR_i(k)$, $k \geq 1$, as the AR model, for the i th coefficient, computed as described above with $\bar{h} = k + 1$. Analogously, $AR(k)$ will indicate the AR model, of order k , for all the coefficients.

C. Kalman-based methods

A disadvantage of the AR models is that their computational complexity and memory requirement, even if linear in n , is also proportional to their order k . Since in the considered system n can be quite large, k should be quite small to reduce the system requirements.

Thus, the next step is that of looking for a state space representation of the turbulent phase, where the state dimension, ν , is hopefully lower than nk . Then the use of a Kalman filter will provide the optimal (with respect to the considered dynamic system) predictions of the state. Several works have been presented in the literature using a Kalman filter approach, [9], [8], [10], [12], [1] just to give some examples. As shown in [1], when the sampling frequency is not too low, a completely decoupled model (i.e. not exploiting cross-correlations between coefficients related to different modes) if properly tuned can take to results (almost) comparable to those obtained with non-decoupled systems.

Consider the following linear dynamic system,

$$\begin{cases} \bar{x}(t+1) = \bar{A}\bar{x}(t) + \begin{bmatrix} 0 \\ u(t+2) \\ 0 \end{bmatrix} + \begin{bmatrix} v(t) \\ 0 \\ 0 \end{bmatrix} \\ y(t) = [C \ 0 \ -(FHD)^\dagger] \bar{x}(t) + w(t) \end{cases} \quad (3)$$

where $\bar{x}(t)$ has the following block structure

$$\bar{x}(t) = \begin{bmatrix} \zeta(t) \\ u(t+1) \\ u(t) \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I & 0 \end{bmatrix},$$

A and C are respectively $\nu \times \nu$ and $n \times \nu$ matrices. Furthermore, $y(t)$ is the measurement vector (as defined in (1)), $\bar{x}(t)$ is the overall state of the dynamic system, $v(t)$ and $w(t)$ are assumed to be zero-mean white noise with covariances Q and R . Finally $\zeta(t)$ is the part of the overall state representing the current value of the turbulent phase.

To reduce the computational complexity of the algorithm, the time-invariant asymptotic Kalman filter is used, i.e. the Kalman gain is constant and set to its asymptotic value \bar{K} . Furthermore, it is simple to prove that in this case $\bar{K} = [K^T \ 0 \ 0]^T$, where $K = APC^T(CPC^T + R)^{-1}$, and P is the solution of the following Algebraic Riccati Equation (ARE): $P = A(P - PC^T(CPC^T + R)^{-1}CPA^T + Q)$. Since the last two blocks of the state vector can be trivially updated, the only non elementary computation in the resulting Kalman predictor is $\hat{x}(t+1|t) = A\hat{x}(t|t-1) + K(y(t) - C\hat{x}(t|t-1) + (FHD)^\dagger u(t))$. Then, similarly to (2), the control law can be written as follows: $u(t) = (FHD)^\dagger A\hat{x}(t-1|t-2)$. Where $\hat{x}(t-1|t-2)$ is the estimate of $x(t-1)$ provided by the Kalman filter given the measurements up to time $t-2$.

To have a feasible algorithm from the computational complexity point of view, typical choices for the parameters

are: $C = I$, $\zeta(t) = x(t)$, A , Q , R and K diagonal with $A(i, i) = \mathbf{E}[x_i(t+1)x_i(t)](\mathbf{E}[x_i(t)x_i(t)])^{-1}$, $Q(i, i) = \mathbf{E}[x_i(t)x_i(t)] - A(i, i)\mathbf{E}[x_i(t)x_i(t)]A(i, i)$, R chosen as the already defined measurement noise covariance matrix but setting to 0 all the elements out of the principal diagonal, and \bar{K} as the solution of the corresponding ARE. Unfortunately, as shown in [1], these choices may take to quite poor performances (due to the way in which the parameters have been computed, i.e. the computed K is optimal for a system which, statistically speaking, does not well approximate the real one). However, differently tuning the parameters seems to ensure quite good performances to the decoupled dynamic system ([1]). Hence the aim of the following paragraphs is that of proposing a proper way to compute the parameters to be then used in the Kalman filter.

For simplicity of notation consider the reconstructed values of the i th coefficient in the modal representation of the turbulent phase, $\xi_i(1), \dots, \xi_i(t)$. Assuming that data can be represented with a state space model, we can write the following equations,

$$\begin{cases} \zeta_i(t+1) = A_i\zeta_i(t) + K_i e_i(t) \\ \xi_i(t) = C_i\zeta_i(t) + e_i(t) \end{cases} \quad (4)$$

where e_i is white noise, and $\{\zeta_i, A_i, C_i\}$ are the blocks corresponding to the i th mode in $\{\zeta, A, C\}$ of system (3).

Then, iterating the above state equation we obtain,

$$\zeta_i(t) = (A_i - K_i C_i)^f \zeta_i(t-f) + \sum_{j=1}^f (A_i - K_i C_i)^{j-1} K_i \xi_i(t-j),$$

and assuming $(A_i - K_i C_i)$ to be stable, f sufficiently large ($f = 25$ in our simulations) and substituting in the output equation we obtain,

$$\xi_i(t) = \sum_{j=1}^f C_i (A_i - K_i C_i)^{j-1} K_i \xi_i(t-j) + e_i(t).$$

Similarly to the AR case, the above equation approximatively formulates $\xi_i(t)$ as a weighted sum of its previous values. However, here we are imposing a structure to the weighting coefficients: This can be viewed as a regularization (thus allowing a sort of de-noising of the sample data).

Minimizing the L_2 norm of e_i , we can use the above equation to estimate the values of $\omega_i(j) = C_i (A_i - K_i C_i)^{j-1} K_i$, $j = 1, \dots, f$. If we choose to use a state ξ_i of dimension 1, then we are trying to fit $\omega_i(\cdot)$ with an exponential, which in some cases may not be a good fit. The choice of whenever the identified parameters are well approximating or not the sample data (or they are over-fitting them) it is in practice a problem of order estimation of the system, and can be determined using standard order estimation criteria (e.g. AIC, MDL). By assumption the order of each of these subsystem is bounded i.e. $\nu_i \leq \bar{\nu}$. Notice that in our simulations the mostly chosen orders are either $\nu_i = 1$ or $\nu_i = 2$.

In fact, we have used a subspace identification approach, the PBSID algorithm, to identify the parameters $\{C_i, A_i, K_i\}$. Since entering in the details of the PBSID is out of scope of this paper, we refer the reader to [5],

[6], and references therein, for a complete description of the algorithm. In the following we will refer to this model as the Kalman-based approach.

V. SIMULATIONS AND DISCUSSION

In this Section we assume to be in VLT-like conditions, e.g. $d = 8$ meter and $n_s = 40$, and we compare the models presented in the previous Section in some simulations.

In all the simulations we consider the atmosphere as formed by six layers, characterized by the following parameters: $v_1 = 5\mathbf{u}$ m/s, $\gamma_1^2 = 0.282$, $h_1 = 0\text{Km}$, $v_2 = 7\mathbf{u}$ m/s, $\gamma_2^2 = 0.192$, $h_2 = 2\text{Km}$, $v_3 = 10\mathbf{u}$ m/s, $\gamma_3^2 = 0.215$, $h_3 = 4\text{Km}$. $v_4 = 15\mathbf{v}$ m/s, $\gamma_4^2 = 0.185$, $h_4 = 8\text{Km}$. $v_5 = 25\mathbf{v}$ m/s, $\gamma_5^2 = 0.074$, $h_5 = 12\text{Km}$. $v_6 = 15\mathbf{u}$ m/s, $\gamma_6^2 = 0.052$, $h_6 = 16\text{Km}$. The sampling rate, f_s , varies in the range $250 \div 1000$ Hz, depending on the simulation. The turbulence has been simulated, at higher resolution than that of the grid \mathbb{L} , using the method described in [4].

We use 5000 samples to estimate the values of the parameters characterizing the control laws. Instead, the number of validation samples is set to 1000. The performances of the AO systems are evaluated in closed-loop using the Strehl Ratio (SR) as performance criterion (as commonly done by the astronomers). Notice that the SR can be well approximated by the following: $\text{SR}(t) \approx \exp(-C_e(t))$.

In Table I we report the Strehl ratios (the mean values over the validation interval) obtained at Zenith direction in five different simulation conditions and for six different AO control models, which correspond to those described in the previous sections. In the following we will ideally divide the 6 control models in 2 groups characterized respectively by “low” computational complexities and “relatively higher” complexities. Notice that the complexities of all the considered algorithms are linear in n (all the models assume decoupled modes), then their differences are due to the different multiplicative coefficients of n (which can be significant in a real AO system).

We consider in the low complexity group the zonal control, the PI (where its gains have been optimized on the training data) and the AR(1) model. The others (AR(5), AR(20) and the Kalman-based model (identified with subspace methods as described in the previous Section)) are referred to as “complex models” (however, we stress the fact that generally speaking these are still computationally very convenient with respect to controllers based on full matrices systems).

The rows in Table I reports (for each of the simulations) respectively: the values of r_0 , L_0 , the number of used spatial bases n , the sampling rate f_s , the measurement noise variance σ_m^2 , the theoretical representation error ϵ_r , the theoretical delay error ϵ_d for zonal control, the theoretical measurement noise error ϵ_n for zonal control, the measured SR for the zonal control, the SR for the PI control, the SR for AR models with three different choices of the order (AR(1), AR(5), AR(20)), the SR for the Kalman-based model and, finally, the state dimension ν of the Kalman-based model.

In both Simulation 1 and 2 we consider $r_0 = 0.2$ m and $L_0 = 20$ m, but we use two different AO balanced systems

TABLE I
AO CLOSED-LOOP PERFORMANCES

	Sim 1	Sim 2	Sim 3	Sim 4	Sim 5
r_0 [m]	0.2	0.2	0.4	0.4	0.4
L_0 [m]	20	20	30	30	30
n	900	750	700	700	700
f_s [Hz]	1000	667	500	500	250
σ_n^2 [rad ²]	0.125	0.2	0.1	0.4	0.1
ϵ_r [rad ²]	0.12	0.21	0.07	0.07	0.07
ϵ_d [rad ²]	0.12	0.21	0.11	0.11	0.34
ϵ_n [rad ²]	0.12	0.19	0.09	0.36	0.09
zonal SR	68.12	54.46	75.67	57.62	59.35
PI SR	70.15	56.66	77.21	68.53	60.08
AR(1) SR	68.76	55.54	76.58	60.79	61.58
AR(5) SR	71.72	59.66	79.47	70.45	70.21
AR(20) SR	75.10	63.61	82.55	75.27	73.53
Kalman SR	74.00	61.78	81.22	72.35	70.35
Kalman ν	994	836	775	732	775

(i.e. $\epsilon_r \approx \epsilon_d \approx \epsilon_n$ for a zonal control): In particular, we evaluate the effect on the performances of simultaneously varying the sampling rate, the measurement noise level and the number of bases (maintaining the system in a balanced condition). The more challenging conditions of Simulation 2 are reflected on the lower Strehl ratios with respect to Simulation 1. The effect is a bit lower on complex models than in the zonal control, but the difference is not very significative.

In Simulation 3 we have a system in approximatively balanced condition with $r_0 = 0.4$ m and $L_0 = 30$ m. While in the first two simulations we considered the effect of simultaneously varying all the sources of error (maintaining the system in balanced conditions), in Simulation 4 and 5 we separately evaluate the effect of increasing, respectively, the measurement noise and the delay. It is apparent that in both the cases the complex models are much more robust than the others: The decrease of their SRs is approximatively half of the SR decrease for the low complexity models.

Talking about the performances of the different methods, it is clear that, due to their extreme simplicity, the PI and AR(1) usually allow small SR improvements with respect to the zonal control. However, the use of higher order AR models or the described Kalman-based model seems to give a significant performance improvement.

In terms of maximum performances, AR(20) gives the best results. However the Kalman-based model provides almost comparable performances. Furthermore it is worth to notice that the computational complexity of AR(20) is typically several times higher than that of the considered Kalman-based model: In fact, the Kalman-based model is computationally comparable to the AR(5) model, however, taking advantage of the de-noising effect due to “data regularization” (using the fitted model), it significantly outperforms the AR(5) model. Furthermore, it gives the advantage of providing information about the dynamic of the system which may be exploited introducing other control strategies.

Finally, we investigated also the case of even higher orders for the AR model, however, in our opinion the pour further

performance improvement does not justify the consequent complexity increase.

VI. CONCLUSIONS

In this paper we have compared some computationally efficient methods for computing a proper control for large AO systems. Most of our attention has been concentrated on intruding and evaluating a Kalman-based approach, where the state-space model has been computed taking advantage of recently developed subspace identification methods.

Then, we compared the proposed Kalman-based approach with other suitable models. The comparison has been performed on a set of VLT simulations, however, the computational efficiency of the considered algorithms ensures their scalability even to larger systems (e.g. the ELT case).

Our results show that the use of computationally cheap methods can be useful to significantly improve the performances of large AO systems. In particular, two approaches seem to be the most successful: the use of (quite) high order AR models, and properly computed Kalman-based models. The choice between the two, should be taken considering both their performances and requirements (in terms of computational complexity and allowable space in memory).

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REFERENCES

- [1] A. Beghi, A. Cenedese, F. Maran, and A. Masiero. A comparison of kalman filter based algorithms for turbulent phase control in an adaptive optics system. In *Proc. of the 47th IEEE Conf. on Decision and Control, CDC08*, pages 1839–1844, Cancun, Mexico, Dec. 2008.
- [2] A. Beghi, A. Cenedese, and A. Masiero. Atmospheric turbulence prediction: a pca approach. In *Proc. of the 46th IEEE Conf. on Dec. and Contr., CDC07*, pages 572–577, New Orleans, USA, Dec. 2007.
- [3] A. Beghi, A. Cenedese, and A. Masiero. A comparison between zernike and pca representation of atmospheric turbulence. In *Proc. of the 46th IEEE Conf. on Decision and Control, CDC07*, pages 561–566, New Orleans, USA, December 2007.
- [4] A. Beghi, A. Cenedese, and A. Masiero. Stochastic realization approach to the efficient simulation of phase screens. *Journal of the Optical Society of America A*, 25(2):515–525, February 2008.
- [5] A. Chiuso. On the relation between cca and predictor-based subspace identification. *IEEE Trans. on Automatic Control*, 52(10):1795–1812, 2007.
- [6] A. Chiuso. The role of vector autoregressive modeling in predictor based subspace identification. *Automatica*, 43(6):1034–1048, 2007.
- [7] R. Conan. *Modelisation des effets de l’echelle externe de coherence spatiale du front d’onde pour l’observation a haute resolution angulaire en astronomie*. PhD thesis, Univ. Nice Sophia Antipolis, 2000.
- [8] B. Le Roux. *Commande optimale en optique adaptative classique et multiconjuguee*. PhD thesis, Univ. de Nice Sophia Antipolis, 2003.
- [9] B. Le Roux, J.M. Conan, C. Kulcsar, and et al. Optimal control law for classical and multiconjugate adaptive optics. *Journal of the Optical Society of America A*, 21(7):1261–1276, 2004.
- [10] C. Petit. *Étude de la commande optimale en OA et OAMC, validation numérique et expérimentale*. PhD thesis, Université Paris 13, 2006.
- [11] F. Roddier. *Adaptive optics in astronomy*. Cambridge univ.press, 1999.
- [12] D. Wiberg, L. Johnson, and D. Gavel. Adaptive optics control of wind blown turbulence via translation and prediction. In *Advances in Adaptive Optics II*, , *Proc. of SPIE*, volume 6272, page 62722X. B. L. Ellerbroek and D. B. Calia, eds., 2006.