Analysis of delay-throughput-reliability tradeoff in a multihop wireless channel for the control of unstable systems

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Abstract—In recent years, a growing interest has been devoted to the performance analysis of control systems in presence of realistic feedback channels, which are constrained in terms of transmission rate, delay and reliability. In this paper we leverage on some recent findings both in the control and information theory domains to deeply analyze the effect of communication non-idealities on the control and performance of unstable stochastic scalar systems. More specifically, we consider a multihop wireless feedback channel for the control of an unstable system and investigate how the delay-throughput tradeoff impacts on the controllability of the system. Results show that control channels with multiple short hops, hence having longer delay but higher throughput, are preferable when the direct link has a low SNR or a high bandwidth is available. Anyway, even in a very simple scenario, it is possible to identify an optimal number of hops that allows to stabilize the largest class of plants.

I. INTRODUCTION

The progress made by wireless communication technologies in terms of transmit rate and reliability are promoting their adoption in scenarios that were traditionally prerogative of wired transmission technologies, such as process control, automotive, and factory automation [1], [2], [3]. Moreover, wireless feedback channels are implicitly used in environmental monitoring and control applications based on wireless sensor networks, where data collected by remote sensors are reported to the control station by means of wireless multihop channels. The recent years have then witnessed a growing interest on the performance analysis of control systems with wireless feedback channels [4], [5].

The problem is of particular interest because, traditionally, control theory and communication theory have been developed mostly independently. In particular, the primary objective of control theory was to develop tools to stabilize unstable plants and to optimize some performance metrics in closed loop, under the assumption that the communication channel between sensors and controller and between the controller and the plant were ideal, i.e. without signal distortion, packet loss or delay. These assumptions, however, do not hold with feedback channels realized by means of practical communication technologies, such as Internet connections or multihop wireless links.

The analysis of control systems wherein the control loops are closed through a real-time network and feedback signals are exchanged in the form of data packets has given rise to a new branch of research, called Networked Control System (NCS). Recent results in this area have revealed the existence of a strict connection between the performance of the controlled plant and the Shannon capacity of the feedback channel. However, this is not sufficient to completely characterize the communication channel from a control perspective [6], [7]. For instance, it has been proved that to stabilize an unstable plant through a control loop, the signal-to-noise ratio (SNR) of the feedback channel must be larger than some threshold depending on the unstable eigenvalues of the plant [8], [9], [10]. Another line of research addressed the problem of stabilizing an unstable plant in presence of a feedback channel prone to random packet losses [11], [12], [13], [14], or rate-limited [15], [16], [17]. A subsequent step has been made to include multiple channel limitations into the model, such as packet loss and quantization [18], [19], which however result in complex optimization problems.

The impact of realistic feedback channels on the solution of an LQG (Linear-Quadratic-gaussian) control problem, which consists in finding the control signal of a Linear time-invariant (LTI) plant that minimizes a Quadratic cost function of the system state, when both the system state and the output signal are affected by Gaussian noise, has been recently investigated in [20], [21]. The analysis led to a stability condition that depends on the packet loss probability, signal to quantization noise ratio (SQNR), and end-to-end delay of feedback channel, which are however considered as independent properties of the feedback channel.

These parameters, however, are clearly interrelated, as, for instance, reducing the erasure probability may require increasing the delay or reducing the information rate across the channel, which is to day, decreasing the maximum achievable SQNR. In this paper, we investigate the effect of such inter-dependencies on the performance of a closed loop LTI control system. More specifically, we consider a discrete-time unstable plant, which is controlled by means of a feedback signal generated by a suitable controller, on the basis of the output signal from the plant. The output signal is transmitted to the controller through a wireless communication channel, which can either be direct or multihop, exploiting in the last case...
a certain number of intermediate relays that store and forward the data packets. Therefore, for a given distance between system output and controller, a larger number of short hops will likely increase the throughput and/or reliability of the communication, at the cost of a larger delivery delay. A higher throughput makes it possible to reduce the quantization noise of the signal transmitted to the controller, thus improving the performance of the controlled system, while a lower packet loss probability reduces the risk for the system to perform with state control information. On the other hand, a large delay will impact on the accuracy of the state estimate of the controller, thus decreasing the performance of the system.

In this paper we shed some light on these complex interactions by studying the performance of the controlled system when varying the number of relays used in the wireless feedback channel and, thus, varying throughput, packet loss rate, and delay in an interdependent manner. Results show that for a given channel in terms of distance, bandwidth and power, the tradeoff between throughput, delay and reliability leads to identify a maximum value of the state gain for which the system is stabilizable.

As a general rule, control channels with multiple short hops, hence having longer delay, but higher throughput, are preferable when the direct link has a low SNR (up to 15 dB in the considered example) or a large bandwidth, and that, even in a very simple scenario it is possible to identify an optimal number of hops, that makes it possible to stabilize the largest class of plants.

The rest of the paper is organized as follows. Sec. II defines the system model and summarizes the main findings of [20] concerning the LQG problems in a NCS framework, in presence of packet losses, transmit rate constraint, and delay. Sec. III describes the interdependences among these three parameters for the multihop wireless feedback channel considered in this study. A proof-of-concept study is presented in Sec. IV, where we analyze the specific case of a multihop connection with equally spaced relays. Finally, Sec. V draws the conclusions.

II. SYSTEM MODEL AND PROBLEM FORMULATION

In this section, we first introduce the LQG problem within the NCS framework, then we formally state the problem, and finally we recall the main findings concerning the conditions on the feedback channel performance required for the stability of the controlled system.

A. LQG problem definition

We consider a plant, modeled as a discrete-time, scalar, LTI system, subject to additive white Gaussian measurement and process noise. More specifically, the state of the system at step \( t \), denoted as \( x_t \), evolves according to the following linear model:

\[
\begin{align*}
x_{t+1} &= a x_t + b u_t + w_t \\
y_t &= c x_t + v_t
\end{align*}
\]

where \( u_t \) and \( y_t \) represent the input and output signals of the plant, respectively, whereas \( w_t \) and \( v_t \) are two independent discrete-time Gaussian white noise processes with variance \( \sigma_w^2 \) and \( \sigma_v^2 \), respectively. Finally, \( a, b \) and \( c \) are the state, input and output coefficients, respectively.

We consider the steady state variance as performance index

\[
J = \limsup_{t \to +\infty} \mathbb{E}[y_t^2].
\]

The objective of the LQG problem is then to minimize \( J \) by means of a suitable control signal \( u_t \), which only depends causally on the output signal \( \{y_m, m \leq t\} \), and possibly on its previous values \( \{u_m, m \leq t-1\} \).

In the NCS framework, the plant output \( y_t \) is not directly accessible to the controller, but must be delivered by means of a suitable transmission scheme, as sketched in Fig. 1.

![Fig. 1. NCS scheme for scalar output plants, where the plant decoder is given by the cascade of a linear state predictor and a state feedback.](image)

In general, the output signal is real, with power

\[
\sigma_y^2 = \mathbb{E}[y_t^2].
\]

Since it is not possible to transfer real numbers over a transmission channel with finite capacity, a certain degree of distortion in the signal \( \tilde{y}_t \) reconstructed at the receiver must be accounted for. Typically, the distortion is due to signal quantization. If quantization is fine enough, the quantization noise \( n_t \) can be effectively modeled as a zero-mean additive random process, with identically distributed uncorrelated samples of power \( \sigma_n^2 = \mathbb{E}[n_t^2] \). In this case, the quantized signal can be transmitted, after suitable source coding, with an information rate

\[
R_q \leq \frac{1}{2} \log(\rho),
\]

where \( \rho \) is the SQNR, given by

\[
\rho = \mathbb{E}[y_t^2]/\sigma_n^2.
\]

Therefore, the SQNR is related to the information rate \( R_q \) of the quantized signal, and increases with it. Since the maximum information rate \( R_q \) is upper limited by the channel code rate \( R_c \), then the SQNR cannot be increased above a certain threshold \( \rho^* \), which depends on \( R_c \). According to the Shannon theorem, the channel code rate can be approached with indefinitely low error probability, provided that the coding time is unlimited. In our system, however, each output sample...
$y_t$ shall be coded within a time slot. Therefore, the closer $R_q$ to $R_c$, the higher the packet loss rate.

The feedback channel will thus comprise analog to digital conversion of $y_t$ and source coding of the corresponding bitstream into packets, channel and transmission over the physical channel. At the receiver, after forward error correction, typically a further frame check is performed to drop packets that have not been successfully corrected (packet erasures). Accepted packets are then decoded to yield the correct digital values.

We model the feedback channel as represented in lower part of Fig. 1, where $n_t$ represents the quantization noise. Assuming that a packet is sent at each $t = 0, 1, \ldots, \gamma_t \in \{0, 1\}$ is a Bernoulli process that models the erasure events ($\gamma_t = 0$), occurring with probability $\varepsilon$ at each packet transmission, independently of previous events. Finally, we assume a transmission/processing delay of $d$ steps between the plant output $y_t$ and the control signal $u_t$. One delay is embedded in the state predictor based on the measurements $h_t$ received up to time $t - 1$. As such, it must be $d \geq 1$, and the delay block $z^{-d-1}$ accounts for the additional encoding/decoding delay.

The input-output relation of the feedback channel model considered in this paper is hence the following

$$h_t = \gamma_t - d + 1(y_{t-d+1} + n_{t-d+1}),$$

which is completely characterized by the parameters $\varepsilon$, $d$, and $\rho^*$, with

$$d \geq 1, \quad P[\gamma_t = 0] = \varepsilon, \quad \rho = E[y_t^2]/\sigma_n^2 \leq \rho^*.$$  

These parameters are clearly related, as, for instance, reducing the erasure probability $\varepsilon$ may require increasing the delay $d$ or reducing the information rate $R_q$, i.e., decreasing the maximum achievable SQNR $\rho$. Therefore some trade-offs are expected in the context of feedback control, since all three terms impact the performance of the closed loop system. The exact form of the relation among these parameters will be discussed in the next section for the case of a multihop wireless channel.

**B. Problem statement**

We restrict our attention to the classical LQG structure for the plant decoder, which is given by the cascade of a linear state estimator and a state feedback, as represented in Fig. 1. The state estimator $\xi_t$ (which uses the data up to time $t-d$) is time-varying since it depends on the sequence $\gamma_t$. In fact, if a packet is not received correctly, i.e. $\gamma_t = 0$, then the estimator updates its state using the model only, while when $\gamma_t = 1$ the estimate is adjusted by a correction term, based on the output innovation. The state feedback module, in turn, will simply return a control signal proportional to the predicted state (see [20] for further details).

In this framework, the objective is to solve the following optimization problem:

$$\min_{G, L} J$$

$$\text{s.t.} \lim_{T \to +\infty} \frac{\sum_{t=0}^T E[y_t^2]}{\sum_{t=0}^T E[n_t^2]} \leq \rho^*$$

The constraint (9) sets an upper bound on the SQNR, which cannot exceed the maximum value $\rho^*$ allowed by the channel code rate.

**C. Conditions for system stability**

The LQG problem defined in the previous section has been thoroughly investigated for scalar plants in [20], where authors have proved that the optimal value $J^*$ of the cost is finite if and only if

$$\frac{1}{1 + \frac{a^2}{\rho}} > 1 - \frac{1}{\alpha^2}. \quad (10)$$

This result sets a necessary and sufficient condition on the packet loss rate $\varepsilon$, SQNR $\rho^*$, and end to end delay $d$ of the feedback channel for the system to be controllable and the cost function $J$ to be reduced to $J^*$. As mentioned, however, the feedback channel parameters cannot be freely set, in general, being strictly interrelated. In the next section, we model such interdependencies for the specific case of a wireless multihop feedback channel and, successively, we apply the model to investigate the impact of the delay-throughput-reliability trade-offs in a regular multihop wireless connection with regards to the controllability of a scalar unstable plant.

**III. MULTIHOP WIRELESS FEEDBACK CHANNEL MODEL**

Consider a scenario where transmitter and receiver are separated by a distance $x_1$. Communication occurs according to a multi-hop store & forward mode, through $d - 1$ relays, placed between transmitter and receiver, as illustrated in Figure 2. For the sake of simplicity, we assume full duplex and mutually orthogonal radio links [22], so that the pipeline effect, i.e., simultaneous transmissions of on different radio links, can be fully exploited.\(^1\)

The packet transmission time over each link is fixed and equal to one slot. Therefore, the end-to-end delay with $d$-hops will be exactly equal to $d$ slots, where $d = 1$ corresponds to the case of single-hop (direct) transmission, which is the baseline of our analysis. Now, let $\Gamma_d(i), R_d(i)$ and $\varepsilon_d(i)$ denote the Signal to Noise ratio, transmission rate, and packet loss rate of the $i$-th hop. It should be clear that by reducing the link distance, $\Gamma_d(i)$ would increase with larger $d$, on average. For instance, if we consider a simple path loss propagation model, and assume the same transmit power on all links, the

\(^1\)Note that this assumption can be relaxed by considering time-division scheduling techniques to avoid mutual interference among the link, at the price of a larger end-to-end delay.
received signal power at a distance \(x\) from the transmitter can be expressed as

\[
P(x) \propto \frac{1}{x^\eta}
\]

where \(\eta\) is the path loss exponent. Let \(p_d(i)\) be the received power at the \(i\)th hop, in case of \(d\)-hop communication, i.e.,

\[
p_d(i) = P(x_d(i)^\eta) = p_1 \left( \frac{x_1}{x_d(i)} \right)^\eta,
\]

where \(x_d(i)\) represents the length of the \(i\)-th link and \(p_1\) is the received power in case of direct communication between system output and controller (i.e., for \(d = 1\)). Accordingly, the Signal to Noise Ratio (SNR) at the \(i\)th receiver will be equal to

\[
\Gamma_d(i) = \Gamma_1 \left( \frac{x_1}{x_d(i)} \right)^\eta
\]

where \(\Gamma_1\) is the SNR over the direct link.

From the SNR we can obtain the capacity of the link, which in turn determines the tradeoff between transmit rate and packet loss probability. To this regard, Polyansky et al. have recently proposed in [23], eq.s (296)-(298), the following approximation for the number of information bits \(R\) that can be delivered by a codeword of \(n\) symbols with error probability no greater than \(\varepsilon\) in AWGN channels:

\[
R^\ast(n, \varepsilon) \simeq n C(\Gamma) - \sqrt{n V(\Gamma)} Q^{-1}(\varepsilon) + \frac{\log(n)}{2},
\]

where \(Q^{-1}\) is the complementary cumulative Gaussian distribution function, whereas \(C(\cdot)\) is the Shannon capacity in bit/channel use and \(V(\cdot)\) is a parameter called channel dispersion that, for AWGN channels, equal

\[
C(\Gamma) = \frac{1}{2} \log(1 + \Gamma),
\]

\[
V(\Gamma) = \frac{\log^2(\varepsilon)}{2} \left( 1 - \frac{1}{(\Gamma + 1)^2} \right),
\]

being \(\Gamma\) the SNR at the receiver. The approximation (14) is proved to be generally pessimistic (underestimate the maximal information rate), but excellent for \(n > 200\) with SNR \(\Gamma > 20\) dB, or \(n > 800\) for lower SNR (\(\Gamma \approx 0\) dB). Overall, the approximation is deemed quite accurate when transmitting at a large fraction of channel capacity, say \(\frac{\log M^\ast(n, \varepsilon)}{n} > 0.8C\).

Now, let \(\rho_d(i)\) denote the SQNR that might be sustained at the \(i\)th link. According to condition (4), the source rate can then be limited to \(\frac{1}{2} \log(\rho_d(i))\). Therefore, using (14), we can bind the SQNR and packet loss rate through the following equation

\[
\log \rho_d(i) \leq n_d(i) C(\Gamma_d(i)) - \sqrt{4V(\Gamma_d(i)) n_d(i)} Q^{-1}(\varepsilon_d(i)) + \log n_d(i)
\]

where \(n_d(i) = 2W_d(i)T\) is the number of channel symbols in the encoded packet on the \(i\)-th hop. Clearly, the SQNR cannot be changed at every hop, since the signal output \(y_t\) is typically quantized and encoded at the first transmission.

\[^2\text{Logarithm is here always intended base 2}\]

\[a_{\max}(\varepsilon, d)\]

\[\text{Fig. 3. Maximum stabilizable gain } a_{\max} \text{ versus packet loss rate } \varepsilon \text{ in each link, for different nr. of hops } d. \text{ The channel parameters are as follows: SNR of the direct } (d = 1) \text{ link } \Gamma_1 = 10\text{dB, path loss exponent } \eta = 3, \text{ codeword length } n = 5.\]

Therefore, the actual SQNR \(\rho_d\) that can be guaranteed through a \(d\) hop connection is given by

\[
\rho_d = \min\{\rho_d(i), i = 1, \ldots, d\}.
\]

The stability condition (10), hence, takes the form

\[
\prod_{i=1}^{d} (1 - \varepsilon_d(i)) \geq \frac{1}{1 + \frac{a^2 d}{\rho_d}} > 1 - \frac{1}{a^2}.
\]

\[\text{IV. \text{Numerical example}}\]

In this section we illustrate the tradeoff between the several quantities that describe the communication channel performance with a numerical example. We fix the distance \(x_1\) between source and destination assume that \(d\)-hop transmission is achieved with uniformly spaced relays, that are placed at a distance \(x_d = x_1/d\) from each other, with the first and last relay placed at the same distance \(x_d\) from the source and destination, respectively.

For the sake of keeping our description simple, we let all the transmitters use the same power and all the receivers have the same noise power. Moreover, we model the channel gain with path loss only, so that all \(d\) hops exhibit the same SNR \(\Gamma_d = \Gamma_1 d^n\), where \(\eta\) is the path loss exponent and \(\Gamma_1\) is the SNR in the case no relays are used and transmission is done on a single hop. For our purposes, a wireless channel is therefore completely described by the triple \((\Gamma_1, n, \eta)\).

For a given wireless channel, different choices of the number of hops \(d\), the SQNR \(\rho\) and the packet loss rate \(\varepsilon\) lead to different classes of stabilizable plants. From (19) such class is composed of all systems whose gain \(a\) is bounded by the value \(a_{\max}\) that satisfies (19) with equality. However, since the choice of \(d, \rho, \varepsilon\) is constrained by (17) one can fix two parameters, optimize the third one by satisfying (17) with equality, and obtain the corresponding \(a_{\max}\). In this example,
we vary $\varepsilon$ and $d$ arbitrarily, then derive the maximum SQNR $\rho_{\max}(\varepsilon, d)$ from (17), and the maximum stabilizable gain $a_{\max}(\varepsilon, d)$.

Figure 3 shows the dependence of $a_{\max}$ on $\varepsilon$ for a few values of $d$. It is seen that for any fixed $d$ there is an optimal $\varepsilon$ that yields the highest $a_{\max}$.

Similarly, Figure 4 shows the dependence of $a_{\max}$ on $d$ for a few values of $\varepsilon$, and again, for each fixed $\varepsilon$ there is an optimal value of $d$. In this case, with $\Gamma_1 = 10$ dB, $n = 5$ and $\eta = 3$, the optimal number of hops is between 1 for high $\varepsilon$ and 4 for $\varepsilon$ as low as $10^{-5}$.

The overall maximum for this channel is given by

$$a_{\max}(\varepsilon, d) = 9.62$$

and is achieved with

$$\varepsilon_{\text{opt}} = 2 \cdot 10^{-3}, \ d_{\text{opt}} = 3, \ \rho_{\text{opt}} = \rho_{\max}(\varepsilon_{\text{opt}}, d_{\text{opt}}) = 82.1 \text{ dB}.\$$

In Figure 5 we provide a contour plot of the overall maximum stabilizable gain for channels with $\eta = 2$, and by varying $\Gamma_1$ and $n$. It is clearly seen that higher SNRs and bandwidths increase the class of stabilizable plants, and that the increase in $a_{\max}$ is approximately exponential in the code length, so that in designing the feedback channel one can trade transmit power for bandwidth. Less obviously, the optimal number of hops is seen to increase slightly with code length, and to decrease with SNR, so that the use of multiple hops is particularly beneficial with low power, but not so much for narrowband channels.

V. CONCLUSIONS AND FUTURE WORK

We have considered the problem of stabilizing a linear unstable plant through a feedback channel that is realistically limited in its communication capabilities. By restricting our focus to scalar systems, we have derived a stabilizability condition that links the plant state gain to the channel throughput, delay, and packet loss rate.

In a wireless environment where the channel is essentially characterized by transmit power and bandwidth, we have considered the possibility of introducing slotted multihop transmission, where in slot $i$ each relay forwards the message he received and decoded in slot $i - 1$. This allows to trade SNR against delay, by appropriately choosing the number of hops and distance between them. Furthermore, by making use of recent results on finite blocklength coding we have been able to relate the packet loss rate with the throughput of the feedback signal and the channel bandwidth.

An example analysis on a simplified scenario has shown that there exists an optimum number of hops which allows to achieve stabilizability of the largest class of plants.

It is left for future work to extend the analysis to a more complex wireless network scenario, and to multidimensional systems.

REFERENCES


