Global asymptotic stability of a PID control system with Coulomb friction

Andrea Bisoffi

University of Trento
Department of Industrial Engineering
Doctoral School in Materials, Mechatronics and Systems Engineering

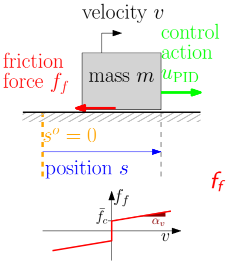
Joint work with Mauro Da Lio, Andrew R. Teel, Luca Zaccarian

Padova, November 25, 2016

Outline

Outline

Model description



$$u_{\text{PID}}(t) := -\bar{k}_p s(t) - \bar{k}_i \int_0^t s(\tau)d\tau - \bar{k}_d \frac{ds(t)}{dt}$$

$$= -\bar{k}_p s(t) - \bar{k}_i e_i(t) - \bar{k}_d v(t),$$

$$f_f(u_{\text{PID}}, v) := \begin{cases} \bar{f}_c \operatorname{sign}(v) + \alpha_v v, & \text{if } v \neq 0 \\ u_{\text{PID}}, & \text{if } v = 0,\ |u_{\text{PID}}| < \bar{f}_c \\ \bar{f}_c \operatorname{sign}(u_{\text{PID}}), & \text{if } v = 0,\ |u_{\text{PID}}| \geq \bar{f}_c \end{cases}$$

$$m\dot{v} = u_{\text{PID}} - f_f(u_{\text{PID}}, v)$$

With $u := \frac{u_{\text{PID}} - \alpha_v v}{m}$, $(k_p, k_v, k_i) := \left(\frac{\bar{k}_p}{m}, \frac{\bar{k}_d + \alpha_v}{m}, \frac{\bar{k}_i}{m}\right)$, $f_c := \frac{\bar{f}_c}{m}$ $\qquad u_{\text{PID}} = m\,u$ for $v = 0$

$$\dot{e}_i = s$$

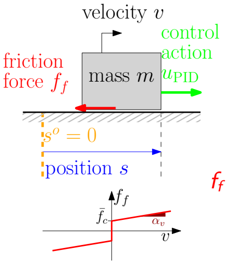
$$\dot{s} = v$$

$$\dot{v} = \begin{cases} u - f_c & \text{if } v > 0 \text{ or } (v = 0, u \geq f_c) \\ 0 & \text{if } (v = 0, |u| < f_c) \\ u + f_c & \text{if } v < 0 \text{ or } (v = 0, u \leq -f_c) \end{cases}$$

$$u = -k_p s - k_v v - k_i e_i,$$

Physical parameters $\bar{k}_p$, $\bar{k}_i$, $\bar{k}_d$, $\bar{f}_c$ vs **normalized** parameters k_p, k_i, k_d, f_c.

Model description



$$u_{\mathrm{PID}}(t) := -\bar k_p s(t) - \bar k_i \int_0^t s(\tau)d\tau - \bar k_d \frac{ds(t)}{dt}$$

$$= -\bar k_p s(t) - \bar k_i e_i(t) - \bar k_d v(t),$$

$$f_f(u_{\mathrm{PID}}, v) := \begin{cases} \bar f_c\,\mathrm{sign}(v) + \alpha_v v, & \text{if } v \neq 0 \\ u_{\mathrm{PID}}, & \text{if } v = 0,\ |u_{\mathrm{PID}}| < \bar f_c \\ \bar f_c\,\mathrm{sign}(u_{\mathrm{PID}}), & \text{if } v = 0,\ |u_{\mathrm{PID}}| \geq \bar f_c \end{cases}$$

$$m\dot v = u_{\mathrm{PID}} - f_f(u_{\mathrm{PID}}, v)$$

With $u := \frac{u_{\mathrm{PID}} - \alpha_v v}{m}$, $(k_p, k_v, k_i) := \left(\frac{\bar k_p}{m}, \frac{\bar k_d + \alpha_v}{m}, \frac{\bar k_i}{m}\right)$, $f_c := \frac{\bar f_c}{m}$ $\qquad u_{\mathrm{PID}} = m\,u$ for $v = 0$

$$\dot e_i = s$$

$$\dot s = v$$

$$\dot v = \begin{cases} u - f_c & \text{if } v > 0 \text{ or } (v = 0, u \geq f_c) \\ 0 & \text{if } (v = 0, |u| < f_c) \\ u + f_c & \text{if } v < 0 \text{ or } (v = 0, u \leq -f_c) \end{cases}$$

$$u = -k_p s - k_v v - k_i e_i,$$

Physical parameters $\bar k_p$, $\bar k_i$, $\bar k_d$, $\bar f_c$ vs **normalized** parameters k_p, k_i, k_d, f_c.

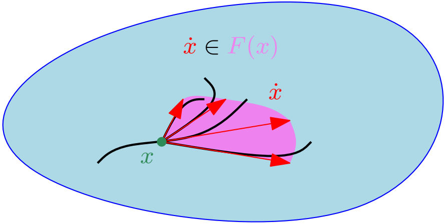
Reformulation with a special differential inclusion

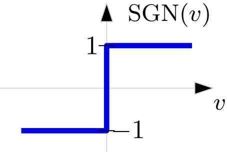
$$\dot{e}_i = s$$
$$\dot{s} = v$$
$$\dot{v} = \begin{cases} u - f_c & \text{if } v > 0 \text{ or } (v = 0, u \geq f_c) \\ 0 & \text{if } (v = 0, |u| < f_c) \\ u + f_c & \text{if } v < 0 \text{ or } (v = 0, u \leq -f_c) \end{cases}$$
$$u = -k_p s - k_v v - k_i e_i,$$

$\Downarrow$[1]

$$\dot{e}_i = s$$
$$\dot{s} = v$$
$$\dot{v} \in -k_i e_i - k_p s - k_v v - f_c \mathrm{SGN}(v)$$

differential inclusions in general





$$\mathrm{SGN}(v) = \begin{cases} \mathrm{sign}(v) & \text{if } v \neq 0 \\ [-1, 1] & \text{if } v = 0 \end{cases}$$

[1] R. I. Leine and N. van de Wouw, *Stability and convergence of mechanical systems with unilateral constraints.* Springer Science & Business Media, 2007.

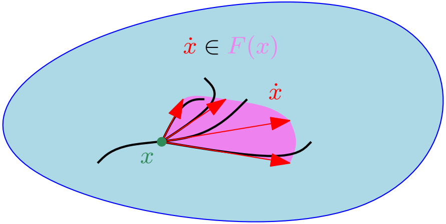
Reformulation with a special differential inclusion

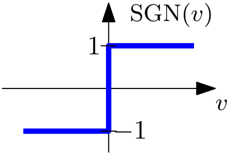
$$\dot{e}_i = s$$
$$\dot{s} = v$$
$$\dot{v} = \begin{cases} u - f_c & \text{if } v > 0 \text{ or } (v = 0, u \geq f_c) \\ 0 & \text{if } (v = 0, |u| < f_c) \\ u + f_c & \text{if } v < 0 \text{ or } (v = 0, u \leq -f_c) \end{cases}$$
$$u = -k_p s - k_v v - k_i e_i,$$

⇓[1]

$$\dot{e}_i = s$$
$$\dot{s} = v$$
$$\dot{v} \in -k_i e_i - k_p s - k_v v - f_c \mathrm{SGN}(v)$$

differential inclusions in general





$$\mathrm{SGN}(v) = \begin{cases} \mathrm{sign}(v) & \text{if } v \neq 0 \\ [-1, 1] & \text{if } v = 0 \end{cases}$$

[1] R. I. Leine and N. van de Wouw, *Stability and convergence of mechanical systems with unilateral constraints.* Springer Science & Business Media, 2007.

Motivation for using this **special** differential inclusion

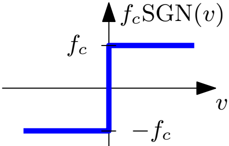
1. The physical model is intuitive, but its discontinuous right hand side makes it hard to prove existence of solutions for each initial conditions and stability properties, whereas the differential inclusion guarantees structurally existence of solutions and lets us adopt Lyapunov tools.
2. No artificial solutions are introduced by the differential inclusion, for which the next Lemma establishes uniqueness of solutions. The unique solution to the diff' incl' must be necessarily the unique sol' to the physical model because the diff' incl' allows more selections for $\dot{v}$ than the phys' model.

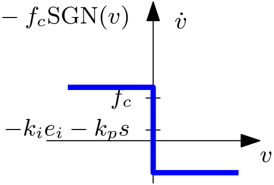
Lemma (solutions are unique and complete)

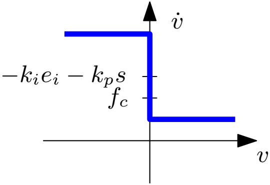
For any initial condition $z(0) \in \mathbb{R}^3$, the differential inclusion has a unique solution defined for all $t \geq 0$.

Intuition behind the lemma:

$$\dot{v} \in -k_i e_i - k_p s - k_v v - f_c \mathrm{SGN}(v)$$







Outline

The standing assumption

System:

$$\dot{z} = \begin{bmatrix} \dot{e}_i \\ \dot{s} \\ \dot{v} \end{bmatrix} \in \begin{bmatrix} s \\ v \\ -k_i e_i - k_p s - k_v v - f_c \mathrm{SGN}(v) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k_i & -k_p & -k_v \end{bmatrix} z - \begin{bmatrix} 0 \\ 0 \\ f_c \end{bmatrix} \mathrm{SGN}(v)$$

Assumption

In the absence of friction ($f_c = 0$), the origin is globally asymptotically stable (GAS). Equivalently,

$$k_i > 0,\ k_p > 0,\ k_v k_p > k_i.$$

Equilibria, attractor and first result

- For $z = (e_i, s, v)$ and

$$\begin{aligned} \dot{e}_i &= s \\ \dot{s} &= v \\ \dot{v} &\in -k_i e_i - k_p s - k_v v - f_c \mathrm{SGN}(v) \end{aligned}$$

  the set of equilibria making $\dot{z} = 0$ are $s = v = 0$ and $|e_i| \leq \frac{f_c}{k_i}$.

- Denote the corresponding set

$$\mathcal{A} := \left\{ (e_i, s, v) : \ s = 0, \ v = 0, e_i \in \left[-\frac{f_c}{k_i}, \frac{f_c}{k_i} \right] \right\}.$$

- Our first main result:

Proposition

Under our Assumption, $\mathcal{A}$ is 1) globally attractive and 2) Lyapunov stable for the differential inclusion.

- $s^o = 0$ for simplicity, but the result is easily generalized to piecewise constant setpoints.

Illustration

- $f_c = 1$ m/s^2

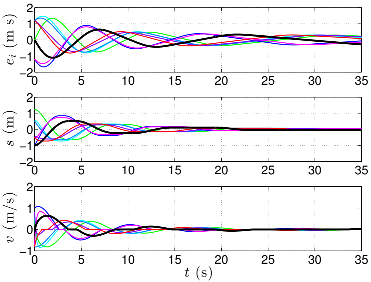
  $(k_v, k_p, k_i) = (6.4, 3, 4)$

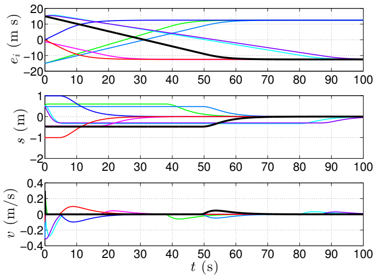
  ➔ complex conjugate roots

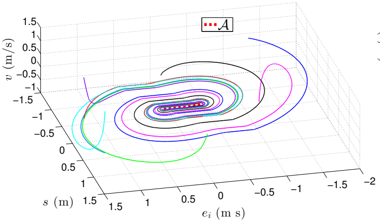
$$\dot{z} \in \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k_i & -k_p & -k_v \end{bmatrix} \begin{bmatrix} e_i \\ s \\ v \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ f_c \end{bmatrix} \mathrm{SGN}(v)$$

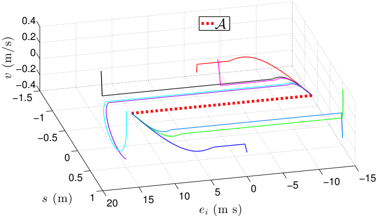
$(k_v, k_p, k_i) = (1.5, 0.66, 0.08)$

➔ three distinct real roots









Discussion about result and literature

The interest in the dynamical properties of friction had its peak in the 1990's.

- modeling direction
  - Dahl model:
    P. R. Dahl, A solid friction model. Tech. Rep. of The Aerospace Corporation El Segundo CA, 1968.
  - models by Bliman and Sorine:
    P.-A. Bliman and M. Sorine, *Easy-to-use realistic dry friction models for automatic control*. Proc. of 3rd European Control Conf., 1995.
  - LuGre model:
    C. Canudas-de-Wit, H. Olsson, K. J. Åström, and P. Lischinsky, *A new model for control of systems with friction*. IEEE Trans. Autom. Control, 1995.
    K. J. Åström and C. Canudas-de-Wit, *Revisiting the LuGre friction model*. Control Systems, IEEE, 2008.
    N. Barahanov and R. Ortega, *Necessary and sufficient conditions for passivity of the LuGre friction model*. IEEE Trans. Autom. Control, 2000.
  - Leuven model:
    J. Swevers, F. Al-Bender, C. G. Ganseman, and T. Projogo, *An integrated friction model structure with improved presliding behavior for accurate friction compensation*. IEEE Trans. Autom. Control, 2000.

Discussion about literature and result

- use of set-valued mapping for the friction force, and hence differential inclusions
  - uncontrolled multi-degree-of-freedom mechanical systems:
    N. van de Wouw and R. I. Leine, *Attractivity of equilibrium sets of systems with dry friction*. Nonlinear Dynamics, 2004.
  - PD controlled 1 d.o.f. system:
    D. Putra, H. Nijmeijer, and N. van de Wouw, *Analysis of undercompensation and overcompensation of friction in 1 DOF mechanical systems*. Automatica, 2007.
  - combination of set-valued friction laws and Lyapunov tools:
    R. I. Leine and N. van de Wouw, *Stability and convergence of mechanical systems with unilateral constraints*. Springer Science & Business Media, 2007.

Discussion about literature and result

- for the same setting (point mass + PID controller + Coulomb and viscous friction) it was proven that no stick-slip limit cycle (so-called hunting) exist:
  - B. Armstrong-Hélouvry and B. Amin, *PID control in the presence of static friction*. Tech. Rep. of Dept. of Elec. Eng. and Computer Science, UW–Milwaukee, 1993.
    B. Armstrong-Hélouvry and B. Amin, *PID control in the presence of static friction: exact and describing function analysis*. Amer. Control Conf., 1994.
    B. Armstrong and B. Amin, *PID control in the presence of static friction: A comparison of algebraic and describing function analysis*. Automatica, 1996.
- the contributions of this work are
  - the proof of GAS of $\mathcal{A}$ — Main result (1)
  - GAS of $\mathcal{A}$ + model regularity $\Rightarrow$ robustness of AS — Main result (2)
    a perturbation of interest is an inflation of ρ_v of SGN;
    $\mathcal{A}$ is globally input-to-state stability (ISS) from ρ_v;
    more gen' friction (Stribeck effect) cause gradual deterioration in ISS sense.

Outline

In order to prove global attractivity and stability:

1. we perform a change of coordinate
2. we define a Lyapunov-like function.

Change of coordinates

- Apply change of coordinates

$$\begin{aligned} \sigma &:= -k_i s \\ \phi &:= -k_i e_i - k_p s \quad \text{to} \quad \dot z := \begin{bmatrix} \dot e_i \\ \dot s \\ \dot v \end{bmatrix} \in \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k_i & -k_p & -k_v \end{bmatrix} z - \begin{bmatrix} 0 \\ 0 \\ f_c \end{bmatrix} \mathrm{SGN}(v) \\ v &:= v \end{aligned}$$

- . . . and get dynamics

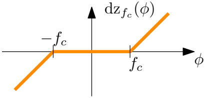
$$\dot x := \begin{bmatrix} \dot\sigma \\ \dot\phi \\ \dot v \end{bmatrix} \in \begin{bmatrix} -k_i v \\ \sigma - k_p v \\ \phi - k_v v - f_c \,\mathrm{SGN}(v) \end{bmatrix} = \begin{bmatrix} 0 & 0 & -k_i \\ 1 & 0 & -k_p \\ 0 & 1 & -k_v \end{bmatrix} \begin{bmatrix} \sigma \\ \phi \\ v \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ f_c \end{bmatrix} \mathrm{SGN}(v)$$
$$= Ax - b\,\mathrm{SGN}(v) =: F(x)$$

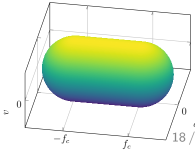
- Attractor

$$\mathcal{A} = \{(\sigma, \phi, v) \colon |\phi| \le f_c, \sigma = 0, v = 0\}$$

- Distance from attractor

$$|x|_{\mathcal{A}}^2 := \big(\inf_{y \in \mathcal{A}} |x - y|\big)^2 = \sigma^2 + v^2 + \mathrm{dz}_{f_c}(\phi)^2$$



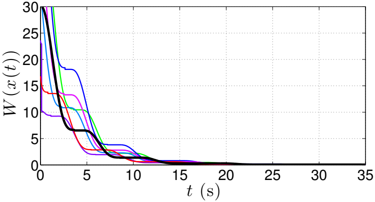


Lyapunov-like function

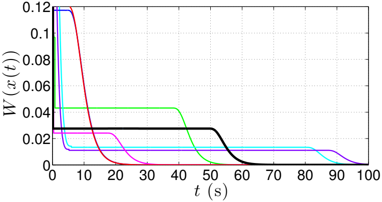
$$W(x) := \begin{bmatrix} \sigma \\ v \end{bmatrix}^T \begin{bmatrix} \frac{k_v}{k_i} & -1 \\ -1 & k_p \end{bmatrix} \begin{bmatrix} \sigma \\ v \end{bmatrix} + \min_{f \in f_c \operatorname{SGN}(v)} |\phi - f|^2$$

$$= \min_{f \in f_c \operatorname{SGN}(v)} \begin{bmatrix} \sigma \\ \phi - f \\ v \end{bmatrix}^T \begin{bmatrix} \frac{k_v}{k_i} & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & k_p \end{bmatrix} \begin{bmatrix} \sigma \\ \phi - f \\ v \end{bmatrix} = \min_{f \in f_c \operatorname{SGN}(v)} \begin{bmatrix} \sigma \\ \phi - f \\ v \end{bmatrix}^T P \begin{bmatrix} \sigma \\ \phi - f \\ v \end{bmatrix}$$

complex conjugate roots



three distinct real roots



Immediate to check:

- ▷ $W(x) = 0$ for all $x \in \mathcal{A}$
- ▷ W is not continuous

for $\{(\sigma_i, \phi_i, v_i)\}_{i=0}^{+\infty} = \{(0, 0, (\frac{1}{2})^i\}_{i=0}^{+\infty}$, W converges to f_c^2 but $W(0) = 0$

Properties of the Lyapunov-like function W

Properties of W

The Lyapunov-like function W is:

1. **lower semicontinuous** (lsc)
2. **lower bounded**:

$$\exists c_1 > 0\colon\ c_1 |x|_{\mathcal{A}}^2 \leq W(x) \quad \forall x \in \mathbb{R}^3$$

3. **decreasing along trajectories**:

$$\exists c > 0\colon \text{ for each sol' } x = (\sigma, \phi, v) \text{ to the diff' incl'},$$

$$\forall t_2 \geq t_1 \geq 0 \quad W(x(t_2)) - W(x(t_1)) \leq -c \int_{t_1}^{t_2} v(t)^2 dt.$$

Lower semicontinuity of W

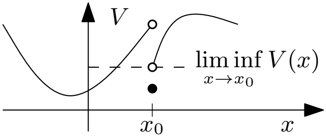
Properties of W

The Lyapunov-like function W is:

1. **lower semicontinuous** (lsc)

- $W: \mathbb{R}^n \to \mathbb{R}$ is lower semicontinuous if

$$\liminf_{x \to x_0} W(x) \geq W(x_0)$$



and $\liminf_{x \to x_0} W(x) = \lim_{\epsilon \to 0} \left(\inf \left\{ W(x) \colon x \in \mathbb{B}(x_0, \epsilon) \backslash \{x_0\} \right\} \right)$

$$= \sup_{\epsilon > 0} \left(\inf \left\{ W(x) \colon x \in \mathbb{B}(x_0, \epsilon) \backslash \{x_0\} \right\} \right).$$

- the proof is merely technical

for $\{(\sigma_i, \phi_i, v_i)\}_{i=0}^{+\infty} = \{(0, 0, (\frac{1}{2})^i\}_{i=0}^{+\infty}$, W converges to f_c^2 but $W(0) = 0$

Lower boundedness of W

Properties of W

The Lyapunov-like function W is:

2. **lower bounded**:

$$\exists c_1 > 0 \colon \; c_1 |x|_{\mathcal{A}}^2 \le W(x) \quad \forall x \in \mathbb{R}^3$$

With $|x|_{\mathcal{A}}^2 := \left(\inf_{y\in\mathcal{A}} |x-y|\right)^2 = \sigma^2 + v^2 + \mathrm{dz}_{f_c}(\phi)^2$,

$$\begin{aligned}
\overbrace{\min\{\mathfrak{g}, 1\}}^{=:c_1} |x|_{\mathcal{A}}^2 \quad & \le \\
\overbrace{\min_{f\in[-f_c, f_c]} (\phi - f)^2}^{=\mathrm{dz}_{f_c}(\phi)^2} + \mathfrak{g}(\sigma^2 + v^2) \quad & \le \\
\min_{f \in f_c \,\mathrm{SGN}(v)} |\phi - f|^2 + \begin{bmatrix} \sigma \\ v \end{bmatrix}^T \begin{bmatrix} \frac{k_v}{k_i} & -1 \\ -1 & k_p \end{bmatrix} \begin{bmatrix} \sigma \\ v \end{bmatrix} \quad & =: W(x)
\end{aligned}$$

Decrease of W

Properties of W

The Lyapunov-like function W is:

3. **decreasing along trajectories**:

$\exists c > 0$: for each sol' $x = (\sigma, \phi, v)$ to the diff' incl',

$$\forall t_2 \geq t_1 \geq 0 \quad W(x(t_2)) - W(x(t_1)) \leq -c \int_{t_1}^{t_2} v(t)^2 dt.$$

Claim → Fact → 3

Decrease of W: Claim

Given $\dot{x} \in Ax - b\,\mathrm{SGN}(v) := \begin{bmatrix} 0 & 0 & -k_i \\ 1 & 0 & -k_p \\ 0 & 1 & -k_v \end{bmatrix} x - \begin{bmatrix} 0 \\ 0 \\ f_c \end{bmatrix} \mathrm{SGN}(v)$,

Claim $\to$ Fact $\to$ 3

consider the three affine systems $\quad (|\xi|_P^2 := \xi^T P \xi)$

Initial value problem		Its Lyap' function
$\dot{\xi} = f_1(\xi) := A\xi - b,$	$\xi(0) = \bar{\xi}_1,$	$W_1(\xi) := \left\| \begin{bmatrix} \sigma \\ \phi - f_c \\ v \end{bmatrix} \right\|_P^2$
$\dot{\xi} = f_0(\xi) := \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xi,$	$\xi(0) = \bar{\xi}_0,$	$W_0(\xi) := \left\| \begin{bmatrix} \sigma \\ 0 \\ 0 \end{bmatrix} \right\|_P^2$
$\dot{\xi} = f_{-1}(\xi) := A\xi + b,$	$\xi(0) = \bar{\xi}_{-1},$	$W_{-1}(\xi) := \left\| \begin{bmatrix} \sigma \\ \phi + f_c \\ v \end{bmatrix} \right\|_P^2$

Claim

There exists $c > 0$ such that, for each initial condition $(\bar{\sigma}, \bar{\phi}, \bar{v})$, one can select $k \in \{-1, 0, 1\}$ and $T > 0$ such that:

1. the solution to $\dot{\xi} = f_k(\xi)$, $\bar{\xi}_k = (\bar{\sigma}, \bar{\phi}, \bar{v})$ satisfies
$$\xi(t) = x(t) \quad \forall t \in [0, T]$$
2. the function W_k satisfies
$$W(\xi(t)) = W_k(\xi(t)) \text{ and } \tfrac{d}{dt} W_k(\xi(t)) \leq -c|\xi_v(t)|^2 \quad \forall t \in [0, T].$$

Decrease of W: Fact

Claim → **Fact** → 3

Fact: a generalized inequality for integrals and Dini derivatives[2]

Given $t_2 > t_1 \geq 0$,

1. $W(x(\cdot))$ is lsc
2. $-c\,v(\cdot)^2$ is locally integrable in $[t_1, t_2]$
3. $D_+W(x(\tau)) \leq -cv^2(\tau), \forall \tau \in [t_1, t_2]$

$$\Rightarrow W(x(t_2)) - W(x(t_1)) \leq \int_{t_1}^{t_2} -cv^2(\tau)d\tau.$$

lower right Dini derivative of h: $D_+h(t) := \liminf_{\epsilon \to 0^+} \frac{h(t+\epsilon)-h(t)}{\epsilon}$

1. ✓ because the composition of a lsc and a continuous function is lsc
2. ✓ because solutions are absolutely continuous
3. ✓ from the claim: for each $(\bar{\sigma}, \bar{\phi}, \bar{v})$ & on nonzero closed intervals
   - x and ξ coincide
   - W and W_k coincide
   - $W(x(\cdot))$ differentiable from the right
   - $D_+W(x(\cdot))$ coincides with right derivative
   - right derivative upper bounded by $-cv^2$.

Claim

1. the solution to $\dot{\xi} = f_k(\xi)$, $\bar{\xi}_k = (\bar{\sigma}, \bar{\phi}, \bar{v})$ satisfies $\forall t \in [0, T]$ $\xi(t) = x(t)$.
2. W_k satisfies $\forall t \in [0, T]$
   $W(\xi(t)) = W_k(\xi(t))$
   $\frac{d}{dt} W_k(\xi(t)) \leq -c|\xi_v(t)|^2$

[2] J. W. Hagood and B. S. Thomson, *Recovering a function from a Dini derivative*, The American Mathematical Monthly, 2006.

Decrease of W: Fact

Claim → **Fact** → 3

Fact: a generalized inequality for integrals and Dini derivatives[2]

Given $t_2 > t_1 \geq 0$,

1. $W(x(\cdot))$ is lsc
2. $-c\,v(\cdot)^2$ is locally integrable in $[t_1, t_2]$
3. $D_+ W(x(\tau)) \leq -cv^2(\tau), \forall \tau \in [t_1, t_2]$

$\Rightarrow W(x(t_2)) - W(x(t_1)) \leq \int_{t_1}^{t_2} -cv^2(\tau)d\tau.$

lower right Dini derivative of h: $D_+ h(t) := \liminf_{\epsilon \to 0^+} \frac{h(t+\epsilon) - h(t)}{\epsilon}$

1. ✓ because the composition of a lsc and a continuous function is lsc
2. ✓ because solutions are absolutely continuous
3. ✓ from the claim: for each $(\bar{\sigma}, \bar{\phi}, \bar{v})$ & on nonzero closed intervals
   - x and ξ coincide
   - W and W_k coincide
   - $W(x(\cdot))$ differentiable from the right
   - $D_+ W(x(\cdot))$ coincides with right derivative
   - right derivative upper bounded by $-cv^2$.

Claim

1. the solution to $\dot{\xi} = f_k(\xi)$, $\bar{\xi}_k = (\bar{\sigma}, \bar{\phi}, \bar{v})$ satisfies $\forall t \in [0, T]$ $\xi(t) = x(t)$.
2. W_k satisfies $\forall t \in [0, T]$

$$W(\xi(t)) = W_k(\xi(t))$$

$$\frac{d}{dt} W_k(\xi(t)) \leq -c|\xi_v(t)|^2$$

[2] J. W. Hagood and B. S. Thomson, *Recovering a function from a Dini derivative*, The American Mathematical Monthly, 2006.

Decrease of W: Fact

Claim → **Fact** → 3

Fact: a generalized inequality for integrals and Dini derivatives[2]

Given $t_2 > t_1 \geq 0$,

1. $W(x(\cdot))$ is lsc
2. $-c\, v(\cdot)^2$ is locally integrable in $[t_1, t_2]$
3. $D_+ W(x(\tau)) \leq -cv^2(\tau), \forall \tau \in [t_1, t_2]$

$\Rightarrow W(x(t_2)) - W(x(t_1)) \leq \int_{t_1}^{t_2} -cv^2(\tau) d\tau.$

lower right Dini derivative of h: $D_+ h(t) := \liminf_{\epsilon \to 0^+} \frac{h(t+\epsilon)-h(t)}{\epsilon}$

1. ✓ because the composition of a lsc and a continuous function is lsc
2. ✓ because solutions are absolutely continuous
3. ✓ from the claim: for each $(\bar\sigma, \bar\phi, \bar v)$ & on nonzero closed intervals
   - x and ξ coincide
   - W and W_k coincide
   - $W(x(\cdot))$ differentiable from the right
   - $D_+ W(x(\cdot))$ coincides with right derivative
   - right derivative upper bounded by $-cv^2$.

Claim

1. the solution to $\dot\xi = f_k(\xi)$, $\bar\xi_k = (\bar\sigma, \bar\phi, \bar v)$ satisfies $\forall t \in [0, T]$ $\xi(t) = x(t)$.
2. W_k satisfies $\forall t \in [0, T]$
   $W(\xi(t)) = W_k(\xi(t))$
   $\frac{d}{dt} W_k(\xi(t)) \leq -c|\xi_v(t)|^2$

[2] J. W. Hagood and B. S. Thomson, *Recovering a function from a Dini derivative*, The American Mathematical Monthly, 2006.

Decrease of W: Fact

Claim → **Fact** → 3

Fact: a generalized inequality for integrals and Dini derivatives[2]

Given $t_2 > t_1 \geq 0$,

1. $W(x(\cdot))$ is lsc
2. $-c\, v(\cdot)^2$ is locally integrable in $[t_1, t_2]$
3. $D_+ W(x(\tau)) \leq -cv^2(\tau), \forall \tau \in [t_1, t_2]$

$$\Rightarrow W(x(t_2)) - W(x(t_1)) \leq \int_{t_1}^{t_2} -cv^2(\tau) d\tau.$$

lower right Dini derivative of h: $D_+ h(t) := \liminf_{\epsilon \to 0^+} \frac{h(t+\epsilon)-h(t)}{\epsilon}$

1. ✓ because the composition of a lsc and a continuous function is lsc
2. ✓ because solutions are absolutely continuous
3. ✓ from the claim: for each $(\bar\sigma, \bar\phi, \bar v)$ & on nonzero closed intervals
   - x and ξ coincide
   - W and W_k coincide
   - $W(x(\cdot))$ differentiable from the right
   - $D_+ W(x(\cdot))$ coincides with right derivative
   - right derivative upper bounded by $-cv^2$.

Claim

1. the solution to $\dot\xi = f_k(\xi)$, $\bar\xi_k = (\bar\sigma, \bar\phi, \bar v)$ satisfies $\forall t \in [0, T]$ $\xi(t) = x(t)$.
2. W_k satisfies $\forall t \in [0, T]$

$$W(\xi(t)) = W_k(\xi(t))$$

$$\frac{d}{dt} W_k(\xi(t)) \leq -c|\xi_v(t)|^2$$

[2] J. W. Hagood and B. S. Thomson, *Recovering a function from a Dini derivative*, The American Mathematical Monthly, 2006.

Connection with the results by Armstrong & Amin

Just proved that:

Properties of W

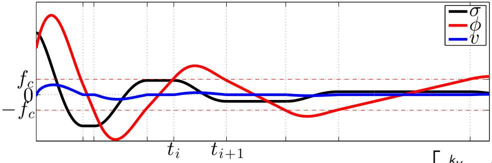
The Lyapunov-like function W is:

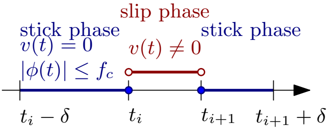
3. **decreasing along trajectories**: $\exists c > 0$: $\forall$sol' $x = (\sigma, \phi, v)$ to the diff' incl',

$$\forall t_2 \geq t_1 \geq 0 \quad W(x(t_2)) - W(x(t_1)) \leq -c \int_{t_1}^{t_2} v(t)^2 dt.$$

In the discussion about literature, we mentioned:

- for the same setting (point mass + PID + Coulomb & viscous friction) it was proven that no stick-slip limit cycle (hunting) exist[3] because $|\sigma(t_{i+1})| < |\sigma(t_i)|$





But $W(x(t_{i+1})) < W(x(t_i))$ and $W(x) := \begin{bmatrix} \sigma \\ v \end{bmatrix}^T \begin{bmatrix} \frac{k_v}{k_i} & -1 \\ -1 & k_p \end{bmatrix} \begin{bmatrix} \sigma \\ v \end{bmatrix} + \min_{f \in f_c \, \mathrm{SGN}(v)} |\phi - f|^2$ imply straightforwardly $\frac{k_v}{k_i}\sigma(t_{i+1})^2 < \frac{k_v}{k_i}\sigma(t_i)^2$.

[3] B. Armstrong and B. Amin, *PID control in the presence of static friction: A comparison of algebraic and describing function analysis*, Automatica, 1996.

Outline

Proof of global attract'y: a general'd invariance principle

Fact: a generalized invariance principle[4]

For $x = (\sigma, \phi, v)$, let $\ell(x) = v^2$. If x is a complete and bounded solution satisfying $\int_0^{+\infty} \ell(x(t))dt < +\infty$, then x converges to the largest forward invariant subset $\mathcal{M}$ of $\Sigma := \{x \in \mathbb{R}^3 : \ell(x) = 0\} = \{x : v = 0\}$.

✓ all x's are complete from lemma uniqueness

✓ all x's are bounded: $\forall t \geq 0$ $\left.\begin{array}{r} W(x(t)) \leq W(x(0)) \\ c_1|x(t)|^2_{\mathcal{A}} \leq W(x(t)) \end{array}\right\} \Rightarrow c_1|x(t)|^2_{\mathcal{A}} \leq W(x(0))$

✓ bounded integral from $c\int_0^t v^2(\tau)d\tau \leq W(x(0)) - W(x(t)) \leq W(x(0))$ & $t \to +\infty$

Properties of W

2. **lower boundedness**: $\exists c_1 > 0$: $c_1|x|^2_{\mathcal{A}} \leq W(x) \quad \forall x \in \mathbb{R}^3$
3. **decrease along trajectories**: $\exists c > 0$: for each sol' $x = (\sigma, \phi, v)$, $\forall t_2 \geq t_1 \geq 0 \quad W(x(t_2)) - W(x(t_1)) \leq -c\int_{t_1}^{t_2} v(t)^2 dt$.

[4] E. P. Ryan, *An integral invariance principle for differential inclusions with applications in adaptive control*, SIAM Journal on Control and Optimization, 1998.

Proof of global attract'y: a general'd invariance principle

Fact: a generalized invariance principle[4]

For $x = (\sigma, \phi, v)$, let $\ell(x) = v^2$. If x is a complete and bounded solution satisfying $\int_0^{+\infty} \ell(x(t))dt < +\infty$, then x converges to the largest forward invariant subset $\mathcal{M}$ of $\Sigma := \{x \in \mathbb{R}^3 \colon \ell(x) = 0\} = \{x \colon v = 0\}$.

✓ all x's are complete from lemma uniqueness

✓ all x's are bounded: $\forall t \geq 0$ $\left.\begin{array}{r} W(x(t)) \leq W(x(0)) \\ c_1|x(t)|^2_{\mathcal{A}} \leq W(x(t)) \end{array}\right\} \Rightarrow c_1|x(t)|^2_{\mathcal{A}} \leq W(x(0))$

✓ bounded integral from $c\int_0^t v^2(\tau)d\tau \leq W(x(0)) - W(x(t)) \leq W(x(0))$ & $t \to +\infty$

Properties of W

2. **lower boundedness**: $\exists c_1 > 0 \colon c_1|x|^2_{\mathcal{A}} \leq W(x) \quad \forall x \in \mathbb{R}^3$
3. **decrease along trajectories**: $\exists c > 0 \colon$ for each sol' $x = (\sigma, \phi, v)$, $\forall t_2 \geq t_1 \geq 0 \quad W(x(t_2)) - W(x(t_1)) \leq -c\int_{t_1}^{t_2} v(t)^2 dt$.

[4] E. P. Ryan, *An integral invariance principle for differential inclusions with applications in adaptive control*, SIAM Journal on Control and Optimization, 1998.

Proof of global attractivity

Fact: a generalized invariance principle

For $x = (\sigma, \phi, v)$, let $\ell(x) = v^2$. If x is a complete and bounded solution satisfying $\int_0^{+\infty} \ell(x(t))dt < +\infty$, then x converges to the largest forward invariant subset $\mathcal{M}$ of $\Sigma := \{x \in \mathbb{R}^3 \colon \ell(x) = 0\} = \{x \colon v = 0\}$.

The largest forward invariant subset $\mathcal{M}$ is $\mathcal{A}$.

- $v = 0$ in $\mathcal{M}$
- $\sigma = 0$ in $\mathcal{M}$: by contradiction each x starting from $v = 0$ and $\sigma \neq 0$ causes a ramp of ϕ that eventually reaches $|\phi| > f_c$ and drives v away from zero, hence out of Σ
- $|\phi| \leq f_c$: otherwise v would become nonzero again.

$$\begin{bmatrix} \dot{\sigma} \\ \dot{\phi} \\ \dot{v} \end{bmatrix} \in \begin{bmatrix} -k_i v \\ \sigma - k_p v \\ \phi - k_v v - f_c \,\mathrm{SGN}(v) \end{bmatrix}$$

Outline

Ingredients for stability

- Stability is proven by finding γ such that

$$|x(t)|_{\mathcal{A}} \leq \gamma |x(0)|_{\mathcal{A}}, \ \forall t \geq 0.$$

- The Lyapunov-like function W is not enough to prove stability because we missed an upper bound

$$c_1 |x|_{\mathcal{A}}^2 \leq W(x) \leq \cancel{c_2 |x|_{\mathcal{A}}^2}.$$

If we had it, we could just write $\forall t > 0$ (using decrease along solutions)

$$c_1 |x(t)|_{\mathcal{A}}^2 \leq W(x(t)) \leq W(x(0)) \leq \cancel{c_2 |x(0)|_{\mathcal{A}}^2}.$$

- However, we can build the two bounds for W in a region R and for an auxiliary function $\hat{W}$ in a $\hat{R}$.

Properties of $\hat{W}$ (and W) in regions $\hat{R}$ (and R)

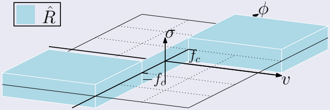
Properties of $\hat{W}$

Define the auxiliary function $\hat{W}$

$$\hat{W}(x) := \tfrac{1}{2}k_1\sigma^2 + \tfrac{1}{2}k_2\big(\mathrm{dz}_{f_c}(\phi)\big)^2 + k_3|\sigma||v| + \tfrac{1}{2}k_4 v^2,$$

and the subsets

$$R := \{x \colon v(\phi - \mathrm{sign}(v)f_c) \geq 0\}$$
$$\hat{R} := \mathbb{R}^3 \backslash R.$$



For suitable $k_1, \ldots, k_4 > 0$ in $\hat{W}$, there exist positive c_1, c_2, $\hat{c}_1$, $\hat{c}_2$ such that

$$c_1|x|^2_{\mathcal{A}} \leq W(x) \leq c_2|x|^2_{\mathcal{A}}, \qquad \forall x \in R,$$
$$\hat{c}_1|x|^2_{\mathcal{A}} \leq \hat{W}(x) \leq \hat{c}_2|x|^2_{\mathcal{A}}, \qquad \forall x \in \hat{R},$$
$$\hat{W}^\circ(x) := \max_{\mathfrak{v}\in\partial\hat{W}(x),\mathfrak{f}\in F(x)} \langle \mathfrak{v}, \mathfrak{f}\rangle \leq 0, \qquad \forall x \in \hat{R},$$

where $\partial\hat{W}(x)$ is the generalized gradient[5] of $\hat{W}$ at x and F is the set-valued mapping of the differential inclusion.

$\dot{V} = \langle \nabla V, f\rangle \leq 0$ for $\dot{x} = f(x)$

[5] F. H. Clarke, *Optimization and nonsmooth analysis*. SIAM, 1990.

Proof of stability

We wanted to prove stability by finding γ such that

$$|x(t)|_{\mathcal{A}} \leq \gamma |x(0)|_{\mathcal{A}}, \ \forall t \geq 0.$$

We prove

$$|x(t)|_{\mathcal{A}} \leq \sqrt{\frac{c_2 \hat{c}_2}{c_1 \hat{c}_1}} |x(0)|_{\mathcal{A}}, \ \forall t \geq 0$$

by splitting into two cases:
Case (i): $x(t) \notin R$, $\forall t \geq 0$.
Case (ii): $\exists t_1 \geq 0$ such that $x(t_1) \in R$.

Proof of stability, Case (i)

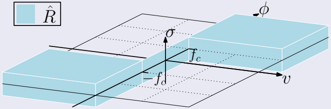
Case (i): $x(t) \notin R$, $\forall t \geq 0$. Then for all $t \geq 0$:

- $x(t) \in \hat{R}$
- $\hat{W}^\circ(x(t)) \leq 0 \Rightarrow \hat{W}(x(t)) \leq \hat{W}(x(0))$ from [6]
- $\hat{c}_1 |x(t)|^2_{\mathcal{A}} \leq \hat{W}(x(t)) \leq \hat{W}(x(0)) \leq \hat{c}_2 |x(0)|^2_{\mathcal{A}}$
- $|x(t)|^2_{\mathcal{A}} \leq \frac{\hat{c}_2}{\hat{c}_1} |x(0)|^2_{\mathcal{A}}$
- $1 \leq \sqrt{c_2/c_1}$
- $|x(t)|_{\mathcal{A}} \leq \sqrt{\frac{c_2 \hat{c}_2}{c_1 \hat{c}_1}} |x(0)|_{\mathcal{A}}$ (claim)

Properties of $\hat{W}$

$$W(x) := \min_{f \in f_c \,\mathrm{SGN}(v)} \begin{bmatrix} \sigma \\ \phi - f \\ v \end{bmatrix}^T P \begin{bmatrix} \sigma \\ \phi - f \\ v \end{bmatrix}$$

$$\hat{W}(x) := \tfrac{1}{2} k_1 \sigma^2 + \tfrac{1}{2} k_2 \left(\mathrm{dz}_{f_c}(\phi)\right)^2 + k_3 |\sigma||v| + \tfrac{1}{2} k_4 v^2$$



For suitable $k_1, \dots, k_4 > 0$ in $\hat{W}$, there exist positive c_1, c_2, $\hat{c}_1$, $\hat{c}_2$ such that

$$c_1 |x|^2_{\mathcal{A}} \leq W(x) \leq c_2 |x|^2_{\mathcal{A}}, \ \forall x \in R$$
$$\hat{c}_1 |x|^2_{\mathcal{A}} \leq \hat{W}(x) \leq \hat{c}_2 |x|^2_{\mathcal{A}}, \ \forall x \in \hat{R}$$
$$\hat{W}^\circ(x) := \max_{\mathfrak{v} \in \partial \hat{W}(x), \mathfrak{f} \in F(x)} \langle \mathfrak{v}, \mathfrak{f} \rangle \leq 0, \ \forall x \in \hat{R}$$

[6] A. R. Teel, L. Praly *On Assigning the Derivative of a Disturbance Attenuation Control Lyapunov Function*. Mathematics of Control, Signals, and Systems, 2000

Proof of stability, Case (ii)

Case (ii): $\exists t_1 \geq 0$ such that $x(t_1) \in R$.

- Consider the smallest $t_1 \geq 0$ such that $x(t_1) \in R$. By cont'ty of sol's $|x(t)|^2_{\mathcal{A}} \leq \frac{\hat{c}_2}{\hat{c}_1}|x(0)|^2_{\mathcal{A}} \forall t \in [0, t_1] \Rightarrow$
  $|x(t_1)|^2_{\mathcal{A}} \leq \frac{\hat{c}_2}{\hat{c}_1}|x(0)|^2_{\mathcal{A}}$
  $|x(t)|^2_{\mathcal{A}} \leq \frac{c_2 \hat{c}_2}{c_1 \hat{c}_1}|x(0)|^2_{\mathcal{A}}, \ \forall t \in [0, t_1]$
- At $t = t_1$
  $W(x(t_1)) \leq c_2 |x(t_1)|^2_{\mathcal{A}} \Rightarrow$
  $W(x(t_1)) \leq c_2 \left(\frac{\hat{c}_2}{\hat{c}_1}|x(0)|^2_{\mathcal{A}}\right)$
- $\forall t \geq t_1$
  $c_1 |x(t)|^2_{\mathcal{A}} \leq W(x(t)) \leq W(x(t_1)) \Rightarrow$
  $c_1 |x(t)|^2_{\mathcal{A}} \leq c_2 \frac{\hat{c}_2}{\hat{c}_1}|x(0)|^2_{\mathcal{A}}, \forall t \geq t_1$

$|x(t)|_{\mathcal{A}} \leq \sqrt{\frac{c_2 \hat{c}_2}{c_1 \hat{c}_1}}|x(0)|_{\mathcal{A}}$ (claim)

Properties of $\hat{W}$

$$W(x) := \min_{f \in f_c \, \mathrm{SGN}(v)} \begin{bmatrix} \sigma \\ \phi - f \\ v \end{bmatrix}^T P \begin{bmatrix} \sigma \\ \phi - f \\ v \end{bmatrix}$$

$$\hat{W}(x) := \tfrac{1}{2} k_1 \sigma^2 + \tfrac{1}{2} k_2 \left(\mathrm{dz}_{f_c}(\phi)\right)^2 + k_3 |\sigma||v| + \tfrac{1}{2} k_4 v^2$$

For suitable $k_1, \ldots, k_4 > 0$ in $\hat{W}$, there exist positive c_1, c_2, $\hat{c}_1$, $\hat{c}_2$ such that

$$c_1 |x|^2_{\mathcal{A}} \leq W(x) \leq c_2 |x|^2_{\mathcal{A}}, \ \forall x \in R$$
$$\hat{c}_1 |x|^2_{\mathcal{A}} \leq \hat{W}(x) \leq \hat{c}_2 |x|^2_{\mathcal{A}}, \ \forall x \in \hat{R}$$
$$\hat{W}^\circ(x) := \max_{\mathfrak{v} \in \partial \hat{W}(x), \mathfrak{f} \in F(x)} \langle \mathfrak{v}, \mathfrak{f} \rangle \leq 0, \ \forall x \in \hat{R}$$

Properties of W

2. $c_1 |x|^2_{\mathcal{A}} \leq W(x)$
3. $\forall t_2 \geq t_1 \geq 0 \quad W(x(t_2)) - W(x(t_1)) \leq -c \int_{t_1}^{t_2} v(t)^2 dt$

Proof of stability, Case (ii)

Case (ii): $\exists t_1 \geq 0$ such that $x(t_1) \in R$.

- Consider the smallest $t_1 \geq 0$ such that $x(t_1) \in R$. By cont'ty of sol's
  $|x(t)|^2_{\mathcal{A}} \leq \frac{\hat{c}_2}{\hat{c}_1}|x(0)|^2_{\mathcal{A}} \forall t \in [0, t_1] \Rightarrow$
  $|x(t_1)|^2_{\mathcal{A}} \leq \frac{\hat{c}_2}{\hat{c}_1}|x(0)|^2_{\mathcal{A}}$
  $|x(t)|^2_{\mathcal{A}} \leq \frac{c_2\hat{c}_2}{c_1\hat{c}_1}|x(0)|^2_{\mathcal{A}}, \ \forall t \in [0, t_1]$•
- At $t = t_1$
  $W(x(t_1)) \leq c_2|x(t_1)|^2_{\mathcal{A}} \Rightarrow$
  $W(x(t_1)) \leq c_2\left(\frac{\hat{c}_2}{\hat{c}_1}|x(0)|^2_{\mathcal{A}}\right)$
- $\forall t \geq t_1$
  $c_1|x(t)|^2_{\mathcal{A}} \leq W(x(t)) \leq W(x(t_1)) \Rightarrow$
  $c_1|x(t)|^2_{\mathcal{A}} \leq c_2\frac{\hat{c}_2}{\hat{c}_1}|x(0)|^2_{\mathcal{A}}, \ \forall t \geq t_1$•

$|x(t)|_{\mathcal{A}} \leq \sqrt{\frac{c_2\hat{c}_2}{c_1\hat{c}_1}}|x(0)|_{\mathcal{A}}$• (claim)

Properties of $\hat{W}$

$$W(x) := \min_{f \in f_c \text{ SGN}(v)} \begin{bmatrix} \sigma \\ \phi - f \\ v \end{bmatrix}^T P \begin{bmatrix} \sigma \\ \phi - f \\ v \end{bmatrix}$$

$$\hat{W}(x) := \tfrac{1}{2}k_1\sigma^2 + \tfrac{1}{2}k_2\left(\mathrm{dz}_{f_c}(\phi)\right)^2 + k_3|\sigma||v| + \tfrac{1}{2}k_4 v^2$$

For suitable $k_1, \ldots, k_4 > 0$ in $\hat{W}$, there exist positive c_1, c_2, $\hat{c}_1$, $\hat{c}_2$ such that

$$c_1|x|^2_{\mathcal{A}} \leq W(x) \leq c_2|x|^2_{\mathcal{A}}, \ \forall x \in R$$
$$\hat{c}_1|x|^2_{\mathcal{A}} \leq \hat{W}(x) \leq \hat{c}_2|x|^2_{\mathcal{A}}, \ \forall x \in \hat{R}$$
$$\hat{W}^\circ(x) := \max_{\mathfrak{v} \in \partial\hat{W}(x), \mathfrak{f} \in F(x)} \langle \mathfrak{v}, \mathfrak{f} \rangle \leq 0, \ \forall x \in \hat{R}$$

Properties of W

2. $c_1|x|^2_{\mathcal{A}} \leq W(x)$
3. $\forall t_2 \geq t_1 \geq 0 \quad W(x(t_2)) - W(x(t_1)) \leq -c\int_{t_1}^{t_2} v(t)^2 dt$

Proof of stability, Case (ii)

Case (ii): $\exists t_1 \geq 0$ such that $x(t_1) \in R$.

- Consider the smallest $t_1 \geq 0$ such that $x(t_1) \in R$. By cont'ty of sol's
  $|x(t)|_{\mathcal{A}}^2 \leq \frac{\hat{c}_2}{\hat{c}_1}|x(0)|_{\mathcal{A}}^2 \forall t \in [0, t_1] \Rightarrow$
  $|x(t_1)|_{\mathcal{A}}^2 \leq \frac{\hat{c}_2}{\hat{c}_1}|x(0)|_{\mathcal{A}}^2$
  $|x(t)|_{\mathcal{A}}^2 \leq \frac{c_2\hat{c}_2}{c_1\hat{c}_1}|x(0)|_{\mathcal{A}}^2, \ \forall t \in [0, t_1]$ •
- At $t = t_1$
  $W(x(t_1)) \leq c_2|x(t_1)|_{\mathcal{A}}^2 \Rightarrow$
  $W(x(t_1)) \leq c_2\left(\frac{\hat{c}_2}{\hat{c}_1}|x(0)|_{\mathcal{A}}^2\right)$
- $\forall t \geq t_1$
  $c_1|x(t)|_{\mathcal{A}}^2 \leq W(x(t)) \leq W(x(t_1)) \Rightarrow$
  $c_1|x(t)|_{\mathcal{A}}^2 \leq c_2\frac{\hat{c}_2}{\hat{c}_1}|x(0)|_{\mathcal{A}}^2, \ \forall t \geq t_1$ •

$|x(t)|_{\mathcal{A}} \leq \sqrt{\frac{c_2\hat{c}_2}{c_1\hat{c}_1}}|x(0)|_{\mathcal{A}}$ • (claim)

Properties of $\hat{W}$

$$W(x) := \min_{f \in f_c \,\mathrm{SGN}(v)} \begin{bmatrix} \sigma \\ \phi - f \\ v \end{bmatrix}^T P \begin{bmatrix} \sigma \\ \phi - f \\ v \end{bmatrix}$$

$$\hat{W}(x) := \tfrac{1}{2}k_1\sigma^2 + \tfrac{1}{2}k_2\left(\mathrm{dz}_{f_c}(\phi)\right)^2 + k_3|\sigma||v| + \tfrac{1}{2}k_4 v^2$$

For suitable $k_1, \ldots, k_4 > 0$ in $\hat{W}$, there exist positive c_1, c_2, $\hat{c}_1$, $\hat{c}_2$ such that

$$c_1|x|_{\mathcal{A}}^2 \leq W(x) \leq c_2|x|_{\mathcal{A}}^2, \ \forall x \in R$$
$$\hat{c}_1|x|_{\mathcal{A}}^2 \leq \hat{W}(x) \leq \hat{c}_2|x|_{\mathcal{A}}^2, \ \forall x \in \hat{R}$$
$$\hat{W}^\circ(x) := \max_{\mathfrak{v} \in \partial\hat{W}(x), \mathfrak{f} \in F(x)} \langle \mathfrak{v}, \mathfrak{f} \rangle \leq 0, \ \forall x \in \hat{R}$$

Properties of W

2. $c_1|x|_{\mathcal{A}}^2 \leq W(x)$
3. $\forall t_2 \geq t_1 \geq 0 \quad W(x(t_2)) - W(x(t_1)) \leq -c\int_{t_1}^{t_2} v(t)^2 dt$

Outline

Stronger AS from $\mathcal{A}$ compact and regularity of $\tilde{F}$

- We have just proved:

Proposition

Under our Assump'n, $\mathcal{A}$ is 1) globally attractive and 2) Lyap' stable for

$$\dot{z} \in \begin{bmatrix} s \\ v \\ -k_i e_i - k_p s - k_v v \end{bmatrix} - f_c \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mathrm{SGN}(v) =: \tilde{F}(z).$$

- $\mathcal{A}$ is compact and (mild) regularity assumptions are satisfied: $\tilde{F}$ has a closed graph, is loc'ly bounded in $\mathbb{R}^3$, and $\tilde{F}(z)$ is convex for every $z \in \mathbb{R}^3$.
- Define:

Definition

The compact set $\mathcal{A}$ is **glob'ly** $\mathcal{KL}$ **AS** if there exist a function $\beta \in \mathcal{KL}$ such that all solutions satisfy $|x(t)|_{\mathcal{A}} \le \beta(|x(0)|_{\mathcal{A}}, t) \quad \forall t \ge 0$.

- Because of $\mathcal{A}$ compact and regularity, $\mathcal{A}$ is glob'ly $\mathcal{KL}$ AS[7].

[7] R. Goebel, R. G. Sanfelice, and A. R. Teel, *Hybrid Dynamical Systems: modeling, stability, and robustness*. Princeton University Press, 2012.

Robustness result for generic perturbation

Proposition

Under our Assump'n, $\mathcal{A}$ is GAS for $\dot{z} \in \begin{bmatrix} s \\ v \\ -k_i e_i - k_p s - k_v v \end{bmatrix} - f_c \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mathrm{SGN}(v) =: \tilde{F}(z)$.

- Perturb the dynamics: take $\rho\colon \mathbb{R}^3 \to \mathbb{R}_{\geq 0}$ satisfying $z \notin \mathcal{A} \Rightarrow \rho(z) > 0$

$$\dot{z} \in \overline{\mathrm{co}}\tilde{F}(z + \rho(z)\mathbb{B}) + \rho(z)\mathbb{B} \qquad \text{(P)}$$

- Define:

Definition

The compact set $\mathcal{A}$ is **robustly glob'ly $\mathcal{KL}$ AS** if there exist a cont'ous function ρ as above such that $\mathcal{A}$ is glob'ly $\mathcal{KL}$ AS for (P).

- Because of $\mathcal{A}$ compact and regularity[8]:

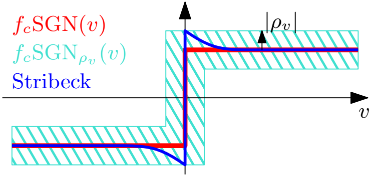
Theorem

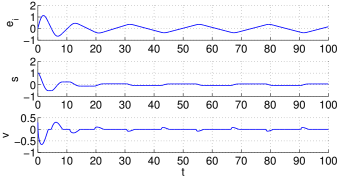
Under our Assumption, $\mathcal{A}$ is robustly globally $\mathcal{KL}$ asymptotically stable for the differential inclusion.

[8] R. Goebel, R. G. Sanfelice, and A. R. Teel, *Hybrid Dynamical Systems: modeling, stability, and robustness*. Princeton University Press, 2012.

Specific perturbation and Stribeck effect

- For constant $\rho_v \in \mathbb{R}$, perturb as: $\dot{z} \in \begin{bmatrix} s \\ v \\ -k_i e_i - k_p s - k_v v \end{bmatrix} - f_c \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mathrm{SGN}_{\rho_v}(v)$





- This perturbation is of interest because it includes the Stribeck effect.
- We have:

Corollary about Stribeck effect

Under our Assumption, the attractor $\mathcal{A}$ is globally input-to-state stable for the perturbed dynamics from input ρ_v.

- Then, the Stribeck effect (➔ persistent oscillations, *hunting*) produces solutions that are (graceful) degradations in the ISS sense of the AS unperturbed solutions.

Outline

Conclusions

So far:

- We characterized the properties of a differential inclusion model of a the feedback interconnection of a sliding mass with a PID controller under Coulomb friction.
- We proved global asymptotic stability of the largest set of closed-loop equilibria.
- Due to the regularity of the differential inclusion, global asymptotic stability was intrinsically robust.
- We proved the ISS of a specific perturbation including the Stribeck effect.

Future work:

- Address the case of static friction larger than Coulomb.
- Propose for that setting hybrid compensation schemes.