Consensus on nonlinear spaces and Graph coloring

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What temperature should we have in this room?

Synchronization = consensus in a vector space

“move towards your neighbors”
What temperature should we have in this room?

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Synchronization = consensus in a vector space

“move towards your neighbors”
Where is the center of the world?

Synchronization on a sphere
Where is the center of the world?

Synchronization on a sphere

Where is the mean position?  How do agents move?
Consensus on nonlinear spaces & Graph coloring

1. Some examples to motivate nonlinear consensus

2. Formalizing consensus on nonlinear spaces

3. Link with graph coloring: (just) a complexity result
I. Orientation synchronization e.g. in formations of spacecraft

State space of orientations = manifold of rotation matrices \( \text{SO}(3) \)

DARWIN interferometer
(NASA / ESA concept study)
II. Coordination on the circle appears in problems involving oscillator networks

Synchronized fireflies
Huygens' clocks

For $\theta_k \in S^1$, $k = 1, 2, \ldots, N$

- phase synchronization: $\theta_1 = \theta_2 = \ldots = \theta_N$
- frequency synchronization: $\frac{d}{dt} \theta_1 = \frac{d}{dt} \theta_2 = \ldots = \frac{d}{dt} \theta_N$
III. Distributed sensor networks e.g. to collect ocean data  (Naomi Leonard et al.)

Autonomous underwater vehicles, sparse communication
Buoyancy-driven at constant speed ~ 40 cm/s

Goal: collective trajectory planning on a simplified AUV model
Agreement on collective motion involves nonlinear spaces

Decision on a direction of straight motion

Synchronization on $S^1$

General motion “in formation”

- translations $\mathbb{R}^2$
- rotations $S^1$

non-trivial coupling: Lie group $SE(2)$
NB: In nonlinear spaces, coordinated motion differs (more difficult) from consensus

Coordinate motion in $\mathbb{R}^n = \text{synchronize velocities in } \mathbb{R}^n$

Motion “in formation”: relative positions of the agents are constant

Equal velocities for all the agents in $T \mathbb{R}^n = \mathbb{R}^n$
NB: In nonlinear spaces, coordinated motion differs (more difficult) from consensus

Coordinate motion on the sphere = ???

The velocities belong to different tangent spaces $TS^n$

The intersection of all tangent spaces is generically empty
NB: In nonlinear spaces, coordinated motion differs (more difficult) from consensus

Algorithms for coordinated motion on Lie groups, see:

“Coordinated motion design on Lie groups”
A. Sarlette, S. Bonnabel and R. Sepulchre,
IV. Coordination on nonlinear spaces is linked to algorithmic applications

- points on a sphere
- lines or subspaces of IR^n (Grassmann manifolds)

Applications: optimal coding, numerical integration, learning of structure in data, optimal placement of converging laser beams / representative planar projections,...
Setting

Identical autonomous agents
  same control law for each agent
  no “leader”, no external supervisor

Limited interconnection links between agents
  agent $k$ has access only to some agents $j$
  interconnection graph $G$ (directed, varying)

Invariance with respect to absolute position
  the agents' behavior only depends
  on their relative positions
Consensus on nonlinear spaces & Graph coloring

1. Some examples to motivate nonlinear consensus

2. Formalizing consensus on nonlinear spaces
   - Synchronization: from vector spaces to the circle
   - Formalization on compact homogeneous manifolds
   - Global synchronization properties

3. Link with graph coloring: (just) a complexity result
A linear algorithm achieves global exponential synchronization on vector spaces

\[
\frac{d}{dt}x_k = \sum_{j \sim k} (x_j - x_k) = d(m_k - x_k)
\]

\[
\begin{cases}
  d = \sum_{j \sim k} 1 \\
  m_k = \frac{1}{d} \sum_{j \sim k} x_j
\end{cases}
\]

For graph G fixed undirected : gradient of

\[
\frac{1}{2} \sum_k \sum_{j \sim k} \|x_j - x_k\|^2
\]
A linear algorithm achieves global exponential synchronization on vector spaces

Exponential synchronization is ensured for any initial condition iff \( G \) is uniformly connected, i.e. \( \exists T \) such that the union of links during \([t, t+T]\) is connected for all \( t \).


For \( G \) undirected: final state = arithmetic mean of the \( x_k(0) \)
This result has two fundamental limitations

The convergence result involves a condition on $G$. But often interconnections depend on the states of the agents.

What about state-dependent graphs?

⇒ under investigation see Bullo et al., Aeyels/De Smet, Blondel/Hendrickx

The global convergence argument does not extend to nonconvex spaces like the circle, sphere,...

How do synchronization algorithms behave globally on manifolds?

⇒ topic of this talk
An algorithm with the same local behavior can be designed on the circle

\[
\frac{d}{dt} \theta_k = \sum_{j \sim k} \sin(\theta_j - \theta_k) = d \, \text{Proj}_{T \mathbb{S}^1}(\theta_k) \left( M_k - e^{i\theta_k} \right)
\]

with \( M_k = \sum_{j \sim k} e^{i\theta_j} \)

Similar to Kuramoto and Vicsek models describing natural behavior

For graph \( G \) fixed undirected: gradient of

\[
\frac{1}{2} \sum_k \sum_{j \sim k} \| e^{i\theta_j} - e^{i\theta_k} \|^2
\]
In the following we will extend this to other “perfectly symmetric” nonlinear spaces

= compact homogeneous manifolds (CCH)

Formally: quotient manifold of a Lie group by a subgroup

Intuitively: “all points are identical”

Examples: sphere $S^n$
rotation matrices $SO(n)$ (and all other compact groups)
Grassmann manifolds (see last part)

In this talk: compact homogeneous manifolds $H$ embedded in $\mathbb{R}^n$ such that $\|x\| = r$ constant for $x \in H$
An alternative distance measure yields convenient properties

**Geodesic distance**

\[ d_g(\theta_k, \theta_j) = |\theta_k - \theta_j| \] on \( S^1 \)

**Chordal distance**

\[ d_c(\theta_k, \theta_j) = 2 \sin \left| \frac{\theta_k - \theta_j}{2} \right| \] on \( S^1 \)

Not obvious on general manifolds

\( d_g^2 \) not smooth everywhere

On CCH manifolds:
consider \( d_c(x_k, x_j) = \|x_k - x_j\| \)

\( d_c^2 \) smooth everywhere
An alternative distance measure yields convenient properties.

**Geodesic distance**

\[ d_g(\theta_k, \theta_j) = |\theta_k - \theta_j| \quad \text{on } S^1 \]

Not obvious on general manifolds

\[ d_g^2 \text{ not smooth everywhere} \]

**Chordal distance**

\[ d_c(\theta_k, \theta_j) = \| e^{i\theta_k} - e^{i\theta_j} \| \quad \text{on } S^1 \]

On CCH manifolds:

Consider \[ d_c(x_k, x_j) = \| x_k - x_j \| \]

\[ d_c^2 \text{ smooth everywhere} \]
An alternative distance measure yields convenient properties

**Geodesic distance**

\[ d_g(\theta_k, \theta_j) = |\theta_k - \theta_j| \quad \text{on } S^1 \]

Not obvious on general manifolds

\[ d_g^2 \text{ not smooth everywhere} \]

**Chordal distance**

\[ d_c(\theta_k, \theta_j) = \|e^{i\theta_k} - e^{i\theta_j}\| \quad \text{on } S^1 \]

On CCH manifolds:

consider \( d_c(x_k, x_j) = \|x_k - x_j\| \)

\[ d_c^2 \text{ smooth everywhere} \]
The “induced arithmetic mean” of the chordal distance is easily computable

Induced arithmetic mean

\[ M = \min_{x \in H} \left( \sum_k d_c(x, x_k)^2 \right) = \text{Proj}_H \left( m = \frac{1}{N} \sum_k x_k \right) \]

\[ \neq \text{traditional Karcher (or Fréchet) mean} = \min_{x \in H} \left( \sum_k d_g(x, x_k)^2 \right) \]

Anti-M

\[ = \max_{x \in H} \left( \sum_k d_c(x, x_k)^2 \right) = \text{Proj}_H (-m) \]
The “induced arithmetic mean” of the chordal distance is easily computable

On $S^1$: $M = \arg \left( \sum_k e^{i\theta_k} \right)$

On $SO(n)$: $M = \text{polar decomposition of } m$

On the Grassmann manifold, representing an element of $Gr(p,n)$ by the orthogonal projection matrix $\Pi_k$ on the corresponding subspace:

$M = p$-dimensional principal eigenspace of $m = \sum \Pi_k$
The induced arithmetic mean allows to define several specific configuration types

**Synchronization** \( x_j = x_k \) for all \( j,k \)

**Consensus** each agent \( k \) moves as close as possible to its fixed neighbors, such that
\[
\forall k, \quad x_k \in M(\{x_j : j \sim k\})
\]

**Anti-Consensus** each agent \( k \) moves as far as possible to its fixed neighbors, such that
\[
\forall k, \quad x_k \in Anti-M(\{x_j : j \sim k\})
\]

**Balancing** each point on the manifold is equally close to the agents, i.e. \( M(\{x_k\}) = H \)
The gradient of $V_G$ yields consensus algorithms

$$\frac{d}{dt} x_k = -\alpha \ \text{grad}_{H,k}(V_\Gamma) \quad \text{for} \quad k = 1, 2, \ldots, N$$

with $\alpha > 0$ for consensus, $\alpha < 0$ for anti-consensus

$$\Rightarrow \quad \frac{d}{dt} x_k = \alpha \ \text{Proj}_{TH}(x_k) \left( \sum_{\{j : j \sim k \text{ or } k \sim j\}} (x_j - x_k) \right)$$

Final algorithm (not gradient for directed, varying graphs)

$$\frac{d}{dt} x_k = \alpha \ \text{Proj}_{TH}(x_k) \left( \sum_{j \sim k} (x_j - x_k) \right)$$

explicit forms on $SO(n)$, Grassmann,...
These developments can be adapted to more complex agent dynamics

“Cascade” approach
use the result of the consensus algorithm as desired velocity, function of the relative positions of the agents, at the input of a tracking algorithm

Planning (consensus) → Tracking (dynamics)

“Energy shaping” approach
for a mechanical system, use $V_\Gamma$ as artificial potential combined with appropriate artificial dissipation
Consensus on nonlinear spaces & Graph coloring

1. Some examples to motivate nonlinear consensus

2. Formalizing consensus on nonlinear spaces
   Synchronization: from vector spaces to the circle
   Formalization on compact homogeneous manifolds
   Global synchronization properties

3. Link with graph coloring: (just) a complexity result
Synchronization is ensured locally. The global behavior is a priori unclear.

Contraction arguments hold if all agents are in a semicircle

Convergence? What is the mean of $\theta_k(0)$?
Fixed but directed graphs can lead to limit cycles, quasi-periodic behavior,...
Undirected graphs ensure convergence to an equilibrium set, but which one?

Some graphs feature stable local attraction equilibria ≠ synchronization

\[ \frac{d}{dt} x_k = -\sum_{j \sim k} (x_j - x_k) \text{ on } \mathbb{R}^n \]

Agents drive away to infinity

\[ \frac{d}{dt} \theta_k = -\sum_{j \sim k} \sin(\theta_j - \theta_k) \text{ on circle} \]

Stable equilibria are not trivial
The existence of local equilibria is sensitive to the attraction profile between connected agents.

Circle:
\[
\frac{d}{dt}\theta_k = \sum_{j \sim k} g(\theta_j - \theta_k)
\]

IAM gradient:
\[
g(\theta) = \sin(\theta)
\]

Synchronization is only stable equilibrium for any fixed undirected graph.

Variation 1

Stable equilibrium different from synchronization even for all-to-all graph.
Alternative algorithms can overcome spurious local equilibria of standard consensus motion

**Gossip algorithm**  = forced asynchrony

At each time, select a single link, and only its 2 agents move towards each other

**Thm:** If G is uniformly connected, synchronizes with probability 1 also on the circle, sphere,...
Alternative algorithms can overcome spurious local equilibria of standard consensus motion

**Auxiliary variables** (can be written with agent-based coordinates)

Embed the manifold in vector space $\mathbb{R}^n$
Assign an auxiliary variable $y_k \in \mathbb{R}^n$ to each agent
The $y_k$ reach agreement by consensus in $\mathbb{R}^n$

Positions $x_k \in H$ are made to follow the projection of $y_k$ on $H$
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Consensus algorithms seem much harder to analyze on nonlinear spaces

Attractive agents, fixed undirected interaction graph
⇒ seems difficult to say if synchronization is the only stable equilibrium

How hard can equilibrium characterization be?

“Consensus on nonlinear spaces and graph coloring”
A. Sarlette, CDC Orlando, pp. 4885-4890 (2011)
Idea: interacting agents setting graph coloring

Graph theory
many complexity results

Graph parametrizes interacting agents in continuous dynamics

Analog computation:
continuous dynamics solve computational problem
Result: equilibrium characterization on projective space is NP-hard

- Graph $k$-coloring
  - NP-hard for $k > 2$

- (Robust) repulsion on projective space $\mathbb{P}^{k-1}_R$
Given graph $G(V,E)$ and integer $k$, find $\varphi : V \rightarrow \{1,2,...,k\}$ (colors)
s.t. $\varphi(a) \neq \varphi(b)$ for all $(a,b) \in E$

Ex. country maps, Sudoku, etc.
Given graph \( G(V,E) \) and integer \( k \), find \( \varphi : V \rightarrow \{1,2,...,k\} \) (colors) s.t. \( \varphi(a) \neq \varphi(b) \) for all \( (a,b) \in E \)

Ex. country maps, Sudoku,...

**Complexity**

For \( k=2 \): \( G \) is 2-colorable \( \iff \) \( G \) is bipartite (polynomial)

For \( k>2 \): NP-hard (in \( \#V \)) to determine if \( G(V,E) \) is \( k \)-colorable
Graph $k$-coloring & Directions in $\mathbb{R}^k$

$k$ different equivalent colors
{1,2,...,k}

$k$ orthogonal lines of $\mathbb{R}^k$
Lines of $\mathbb{R}^k$ define the **projective space** $\mathbb{P}^{k-1}\mathbb{R}$

$x \in \mathbb{P}^{k-1}\mathbb{R}$ represents a line of $\mathbb{R}^k$

Handy representation:

orthonormal projection $\Pi$ onto line $x$

$\Pi \in \mathbb{R}^{k \times k}$, $\text{rank}(\Pi)=1$, $\text{trace}(\Pi)=1$

$\Pi = vv^T / (v^Tv)$

Chordal distance on $\mathbb{P}^{k-1}\mathbb{R}$

$$d_c(\Pi_1, \Pi_2) := \|\Pi_1 - \Pi_2\|_F$$

$$= \sqrt{2 - 2(v_1^Tv_2)^2}$$

$$= \sqrt{2 \sin^2(\phi)}$$
Repulsive agents try to maximize their mutual distance

Cost function

\[ W = \sum_{(a,b) \in E} g(d_c(\Pi_a, \Pi_b)^2) \]

with \( g(x) \) a strictly monotonically increasing function on \([0, 2]\)

Gradient dynamics

\[ \frac{d}{dt} \Pi_a = \text{grad}_{\Pi_a} W \]
\[ = -\sum g'(d_c(\Pi_a, \Pi_b)^2) (\Pi_a \Pi_b \Pi_a^\perp + \Pi_a^\perp \Pi_b \Pi_a) \]

= anti-consensus motion on projective space
Goal: relate the stable equilibria to graph coloring solutions

Stable equilibria = local maxima of $W$

Result about complexity of characterizing stable equilibrium set
(as complex as deciding graph coloring)

Possibility to solve graph coloring by swarm optimization?
(continuous evolution of the swarm converges to solution
= distributed analog computation)
Two particular sets in $P^{k-1}R$

\[ S_o = \{ (\Pi_1, \Pi_2, \ldots, \Pi_N) \in (\mathbb{P}^{k-1} \mathbb{R})^N : \Pi_a \Pi_b = \Pi_b \Pi_a \quad \forall \ a, b \} \]

all states belong to a discrete set of $k$ orthogonal lines = “colors”

\[ S_p(G) = \{ (\Pi_1, \Pi_2, \ldots, \Pi_N) \in (\mathbb{P}^{k-1} \mathbb{R})^N : \Pi_a \Pi_b = 0 \quad \forall \ (a, b) \in E \} \]

every edge is stretched to the maximum distance

Properties :

- $S_p(G)$ can be empty depending on $G$
- $S_p(G)$ global maxima of $W$ if $\neq \emptyset$ \hspace{1cm} (*)
- $S_o \cap S_p(G) \neq \emptyset$ if and only if $G$ is $k$-colorable \hspace{1cm} (**)
Question: Given $G(V,E)$ and $P^{k-1}R$, is any point in $S_0$ a stable equilibrium for the repulsive agents?

Yes/no question (typical decision problem)
about specific property of equilibrium set

**Theorem:** This question is as difficult as graph coloring
-- that is NP-hard for $k > 2$ --
if $g(x)$ satisfies

$$g'(0)/g'(2) > \left\lfloor \frac{N}{k} \right\rfloor / (\left\lfloor \frac{N}{k} \right\rfloor - 1)$$
The condition on coupling function $g(x)$ is not too restrictive ...

Condition \[ \frac{g'(0)}{g'(2)} > \left\lfloor \frac{N}{k} \right\rfloor / (\left\lfloor \frac{N}{k} \right\rfloor - 1) \]

Large class of $g(x)$ coupling functions

Allows $g(x) \propto \text{identity}$ (canonical consensus) for $N/k \to \infty$
Proof idea

- $S_p(G)$: always stable
- $S_o$: ensure unstable
- not empty $\iff$ graph is $k$-colorable
- $\Rightarrow$ stable eq. in $S_o$ $\iff$ graph $k$-colorable

Role of the condition on $g(x)$
Simulations with \( g(x) = \tan(x/2) \) for \( k=3 \)

Petersen graph, 3-colorable
Simulations with $g(x) = \tan(x/2)$ for $k=3$

Grötzsch graph, not 3-colorable
Can we use the distributed dynamical system to solve graph coloring?

Stable equilibria = local maxima of $W$

result about *complexity* of characterizing stable equilibrium set
(as complex as deciding graph coloring)

possibility to solve graph coloring by *swarm optimization*?
(continuous evolution of the swarm converges to solution
 = distributed analog computation)
Global maxima of $W$ in $S_o \cap S_p(G) \leftrightarrow$ graph $k$-coloring

Multi-agent system for
\textbf{colorable} $G$ converges to $S_p(G) \neq S_o \cap S_p(G)$

\textbf{Kochen-Specker Theorem]
There exist \textbf{non-colorable} $G$
for $k=3$ \textbf{with $S_p(G) \neq \emptyset$}

\Rightarrow A system that converges to a point in $S_p(G) \setminus S_o$
can correspond to colorable or non-colorable $G$...
The Kochen-Specker theorem discusses fundamentals of quantum measurement.

Element of \( P^{k-1}R \)

\[ \equiv \text{possible result of projective quantum measurement on } R^k \]

Kochen-Specker:

For \( k \geq 3 \), there does not exist a function \( f \) from the set of possible measurement projectors \( P_i \in P^{k-1}R \) to associated measurement results in \{0,1\} such that for every \( \{P_i\} \) that form a physical observable (i.e. that commute and \( \sum P_i = I \)) we have \( \sum f(P_i) = 1 \).

Use: show a contradiction with classical re-interpretations of quantum laws.
Proof:

Constructs an example of $N$ elements of $\mathbb{P}^{k-1}R$, where mutually orthogonal lines are connected in a graph. Then assigning

$$f(\text{color 1}) = 1, \quad f(\text{other colors}) = 0$$

would solve the task if colorable.

They have a counterexample with $N=31$ agents for $k=3$.

$\Rightarrow$ They construct a situation where all pairs of connected agents are orthogonal in $\mathbb{R}^3$, but the graph is NOT 3-colorable.
Conclusion

General geometric interpretation of consensus allows extension to nonlinear spaces

Consensus motion yields more complex global behavior than on $\mathbb{R}^n$
- possible limit cycles, quasi-periodicity,... for directed graphs
- multiple equilibria for undirected graphs depending on precise coupling function & interaction graph
Conclusion

Graph-coloring complexity of consensus on projective space

For a class of repulsion functions (robustly difficult)

Link not bi-directional: provides no solution for graph coloring

Equilibrium stability as key feature to characterize

NP-hard for $k>2$

⇒ leaves open the case $k=2$ corresponding to the circle
  (which seems not trivial, but further unclear how hard)
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