Exploration of Kinematic Optimal Control on the Lie Group SO(3)

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Introduction

Motivation

- We are interested in solving optimal control problems on a Lie group G

$$\min_{u(\cdot)} \int_0^T l(g(\tau), u(\tau))\, d\tau + m(g(T))$$

subject to

$$\dot{g}(t) = f(g(t), u(t))$$
$$g(0) = g_0$$

with $g(t) \in G$, $t \geq 0$, and $u(t) \in \mathbb{R}^m$, $t \geq 0$.

- Constrained kinematic and dynamic motion planning for single and multiple aerial and underwater vehicles is the driving application

- Other possible interesting applications: Optimal transfer in quantum mechanical systems, satellite maneuvering, ...

Motivation (cont'd)

- We recently developed the **extension to Lie groups** (Saccon *et al.*, CDC, 2010) of the **Projection operator** approach to optimization of trajectory functionals proposed in 2002, by Prof. John Hauser,

  Hauser, J., **A Projection Operator Approach to the Optimization of Trajectory Functionals**, 15th IFAC World Congress, 2002

- The projection operator approach is an iterative algorithm to find the solution of a continuous time nonlinear optimal control problem (including state and input constrained problems via a barrier functional approach).

- At each iteration, a continuous-time **quadratic approximation** of the original problem around the current iterate is constructed (this amounts to solving a suitable continuous-time LQ optimal control problem).

- We are developing a series of tests to asset the numerical performance of Lie group projection operator approach and to compare it against standard methods (e.g., based on discretization, local coordinates).

- **The simplest non trivial example of optimal control we could think about is the extension of the (infinite horizon) Linear Quadratic Regulator to the Lie group SO(3), the group of rotational matrices in $\mathbb{R}^3$.**

  This presentation wants to outline our findings for this particular problem.

Notation

- A Lie group is a differentiable manifold with **smooth** group structure. The set $\mathsf{SO}(3) = \{g \in \mathbb{R}^{3\times 3} : g^T g = I, \det(g) = 1\}$ with standard matrix multiplication is the **special orthogonal** group.

- Being a smooth manifold, at each point $g \in SO(3)$ we can attach a tangent space $T_g\mathsf{SO}(3)$ (vectors). The **cotangent space** $T_g^*\mathsf{SO}(3)$ is the set of linear applications $\alpha : T_g\mathsf{SO}(3) \mapsto \mathbb{R}$ (covectors).

- The disjoint union of all tangent spaces forms the the **tangent bundle** $T\mathsf{SO}(3)$ and, similarly, the disjoint union of all cotangent spaces forms the **cotangent bundle** $T^*\mathsf{SO}(3)$.

- The **natural pairing** between a covector $\alpha \in T_g^*\mathsf{SO}(3)$ and a vector $v \in T_g\mathsf{SO}(3)$ is denoted by $\langle \alpha, v \rangle := \alpha(v)$.

- By differentiating "twice" the inner automorphism $I_h g = hgh^{-1}$, one can define a binary operation $[\cdot,\cdot] : T_e\mathsf{SO}(3) \times T_e\mathsf{SO}(3) \to T_e\mathsf{SO}(3)$, the **Lie bracket**.
  The Lie bracket operation turns the tangent space at the identity $T_e\mathsf{SO}(3)$ into a **Lie algebra**, denoted $\mathfrak{so}(3)$. In matrix form, $\mathfrak{so}(3)$ **is the space of skew-symmetric** 3×3 **matrices** and **the Lie bracket is the matrix commutator** $[A, B] = AB - BA$.

Kinematic Optimal Control on SO(3)

The Linear Quadratic Regulator

- Linear Quadratic Regulator (LQR) problem

$$\min_{u(\cdot)} \frac{1}{2} \int_0^\infty \|x(\tau)\|_Q^2 + \|u(\tau)\|_R^2 \, d\tau \,,$$

subject to

$$\dot{x}(t) = Ax(t) + Bu(t) \,, \qquad x(0) = x_0 \in \mathbb{R}^n \,.$$

- A standard method to obtain an asymptotic stabilizing controller.
- The *weighting matrices Q and R* affect the closed loop behavior of the system, and provide *a penalty of the state and input of the system*, respectively.

The problem studied in this work

- We are studying the following problem (where e is the group identity)

$$\min_{u(\cdot)} \frac{1}{2} \int_0^\infty \|g(\tau) - e\|_Q^2 + \|\xi(\tau)\|_R^2 \, d\tau \,,$$

  subject to

$$\dot{g}(t) = \xi(t) g(t) \,, \qquad g(0) = g_0 \in \mathsf{SO}(3) \,,$$

  where ξ is the *spatial* angular velocity of the coordinate frame g

  Not so many papers addressing optimal control with state penalty!

- $\|g - e\|_Q^2 = \mathrm{tr}((g-e)^T Q (g-e))$, a weighted squared Frobenius norm. In particular, for $Q = I$, we simply get $\|g - e\|^2 = 2\mathrm{tr}(e - g)$.

- The main theoretical tool we use is the *Pontryagin's Maximum Principle* for Lie groups (e.g., Jurdjevic, 1997, Chapter 12)

- The *incremental cost*

$$l(g, \xi) = \|g - e\|_Q^2 + \|\xi\|_R^2$$

  has a *unique local minimum* on $\mathsf{SO}(3) \times \mathbb{R}^3$ for $(g, \xi) = (e, 0)$

Pontryagin's Maximum Principle

Pontryagin's Maximum Principle

- For unconstrained optimal control problems, the PMP requires one to form the pre-Hamiltonian function

$$\hat{H}(g,\xi,p) = l(g,\xi) + \langle \mu, f(g,\xi) \rangle = 1/2\,\mathrm{tr}(e-g) + 1/2\,\xi^T R\xi + \left\langle p, \hat{\xi} g \right\rangle \quad (1)$$

where $p \in T^*\mathrm{SO}(3)$ is the *adjoint* state.

- Then, one defines the Hamiltonian $H : T^*\mathrm{SO}(3) \to \mathbb{R}$

$$H(g,p) = \min_{\xi} \hat{H}(g,\xi,p)$$

with associated optimal control

$$\xi^*(g,p) = \arg\min_{\xi} \hat{H}(g,\xi,p)\,.$$

- The PMP states that, for extremal trajectories, the state and adjoint variables must satisfy the Hamiltonian equations

$$\dot{g} = \frac{\partial H}{\partial p}\,, \qquad \dot{p} = -\frac{\partial H}{\partial g} \quad (2)$$

with **suitable boundary (aka transversality) conditions**.

Pontryagin's Maximum Principle on Lie groups

- It is possible to define a diffeomorphism between $T^*\mathrm{SO}(3)$ and the direct product $\mathrm{SO}(3) \times \mathfrak{g}^*$, i.e., *the bundle is* trivial.
- The diffeomorphism is constructed using $p = (TR_{g^{-1}})^*\mu$ where $p \in T^*\mathrm{SO}(3)$ and $\mu \in \mathfrak{g}^*$.
- Main tool in the trivialization: $\langle p, v_g \rangle_{TG} = \langle p, TR_g\xi \rangle = \langle (TR_g)^* p, \xi \rangle = \langle \mu, \xi \rangle_{\mathfrak{g}}$
- Equivalent necessary conditions for optimality can be obtained using a **right-trivialized version of the Hamiltonian equations**.
- The right-trivialized pre-Hamiltonian $H^+ : \mathrm{SO}(3) \times \mathfrak{g}^* \to \mathbb{R}$ is defined as

$$\hat{H}^+(g, \xi, \mu) := \hat{H}(g, \xi, p)|_{p=(TR_{g^{-1}})^*\mu}$$

where $\mu \in \mathfrak{so}^*(3)$ is the right-trivialized adjoint variable.

Pontryagin's Maximum Principle on Lie groups (cont'd)

- For our problem

$$\hat{H}^+(g,\xi,\mu) = l(g,\xi) + \langle \mu, \xi \rangle = 1/2\,\mathrm{tr}(e-g) + 1/2\,\xi^T R\xi + \langle \mu, \xi \rangle$$

Minimizing the pre-Hamiltonian $\hat{H}^+$ with respect to the input ξ, we obtain the *right-trivialized Hamiltonian*

$$H^+(g,\mu) = \min_{\xi} \hat{H}^+(g,\xi,\mu) = 1/2\,\mathrm{tr}(e-g) - 1/2\,\mu^T R^{-1}\mu$$

where the associated optimal control is

$$\xi^*(g,\mu) = \arg\min_{\xi} H^+(g,\xi,\mu) = -R^{-1}\mu\,.$$

Pontryagin's Maximum Principle on Lie groups (cont'd)

- The PMP requires the optimal state-adjoint trajectory to satisfy the following right-trivialized Hamiltonian equations

$$\dot{g}g^{-1} = \frac{\partial H}{\partial \mu}^{+}$$

$$\dot{\mu} = -\mathrm{ad}^{*}_{\partial H^{+}/\partial \mu}\, \mu - (TR_g)^{*}\, \frac{\partial H}{\partial g}^{+}$$

with boundary conditions $g(0) = g_0$ and $\lim_{T\to\infty} \mu(T) = 0$.

- For our problem, one sees that

$$(TR_g)^{*} \frac{\partial H}{\partial g}^{+} = w(g)$$

where $\widehat{w}(g) = (g - g^T)/2$.

- The right-trivialized Hamiltonian equations describe a mechanical system:

$$\dot{g}g^{-1} = -(R^{-1}\,\mu)^{\wedge}\,,$$
$$\dot{\mu} = (-R^{-1}\,\mu) \times \mu - w(g)\,.$$

Obtained results

Scalar control weighting $R = rI$

- For the special case $Q = I$ and $R = rI$, $r > 0$, $r \in \mathbb{R}$, we can obtain **explicit expressions** for the value function and optimal feedback

- Define

$$\Pi := \{g \in \mathsf{SO}(3) : g = \exp(\pi\hat{n}), n \in \mathbb{R}^3, \|n\| = 1\}.$$

  The set Π is the set of all rotation matrices which define a rotation of π radians about some axis.

- The value function is

$$V(g) = 2\sqrt{r}(2 - \sqrt{1 + \mathrm{tr}(g)}).$$

  Note that $V(g)$ is continuous on $SO(3)$ but it is *not differentiable on* Π.

- Minimum value attained at $g = e$, where $V(g) = 0$.

- Maximum value attained at $g \in \Pi$, where $V(g) = 4\sqrt{r}$.

Scalar control weighting $R = rI$ (cont'd)

- Recall that $\xi^*(t) = -R^{-1}\mu(t)$ and $w(g) := (g - g^T)^\vee/2$
- The optimal control is

$$\xi^*(g) = -\frac{1}{r}\mu_s(g) = -\frac{2}{\sqrt{r}}\frac{w(g)}{\sqrt{1+\text{tr}(g)}}.$$

- The optimal control is a function of (right-trivialized) adjoint variable μ.
- But $\mu(t) = \mu_s(g(t))$ is a function of the state!
- As we will soon explain, $g(t) \to 0$ and $\mu(t) \to 0$ for $t \to \infty$: **The trajectory $(g(t), \mu(t))$, $t \geq 0$, lives in the stable manifold of the equilibrium point $(e, 0) \in G \times \mathfrak{g}^*$.**

Scalar control weighting $R = rI$ (cont'd)

- This fact (explained in details in the paper Saccon *et al.*, 2010, NOLCOS) is due to the existence of a stable *Lagrange submanifold* for the Hamiltonian equations, passing through the point $(e, 0)$ in $T^*\mathrm{SO}(3)$.
- $(e, 0) \in \mathrm{SO}(3) \times \mathfrak{g}^*$ is a **hyperbolic equilibrium point**! Stable and unstable manifolds are present.
- $(e, 0) \in T^*\mathrm{SO}(3)$ is also **hyperbolic** for the (non-trivialized) Hamiltonian equations
- A **Lagrangian submanifold** of a Hamiltonian system of dimension $2n$ is a submanifold of dimension n in which the symplectic form vanishes
- In (Van der Schaft, 91) it is shown that the stable manifold of an hyperbolic equilibrium point is Lagrangian.
- The stable submanifold $\{(g, p_s(g)) \in T^*\mathrm{SO}(3) | g \in \mathrm{SO}(3)/\Pi\}$ is the graph of the 1-form $p_s(g)$.
- A 1-form is closed if and only if it is a graph of a Lagrangian submanifold (Abraham and Marsden, 87)
- as $\mathrm{SO}(3)/\Pi$ is simply connected, $p_s(g)$ is **exact**.
- It should not be surprising that $p_s(g) = \partial V(g)/\partial g$.

Scalar control weighting $R = rI$ (cont'd)

- Note that this is a standard fact: In the standard LQR, the optimal control satisfies $u^*(t) = -R^{-1}B^T p(t) = -R^{-1}B^T Px(t)$, with P the stabilizing solution of the Riccati equation.
- In the linear case the Lagrangian submanifold is just the stable subspace $(x, Px) \in \mathbb{R}^n \times \mathbb{R}^n$, $x \in R^n$.
- **Returning to our problem...**
- We have also showed that $V(g) = 2\sqrt{r}(2 - \sqrt{1 + \text{tr}(g)})$ is the **viscosity solution** of the associated HJB equation.
- For a *infinite* horizon optimal control problem, the HJB equation is $\max_\xi -\hat{H}(g, \xi, DV(g)) = -H(g, DV(g)) = 0\,.$
- A function $u(\cdot)$ is a viscosity solution of $-H(g, Du(g)) = 0$ iff

$$
\begin{aligned}
-H(g,p) &\leq 0 \qquad && \forall p \in D^+u(g)\,,\\
-H(g,p) &\geq 0 \qquad && \forall p \in D^-u(g)\,,
\end{aligned}
$$

  where $D^+u(g) \subset T^*\text{SO}(3)$ and $D^-u(g) \subset T^*\text{SO}(3)$ are **super-** and **sub-differential** of $u(\cdot)$.
- Understanding this on a Lie groups is a **"little"** tricky...

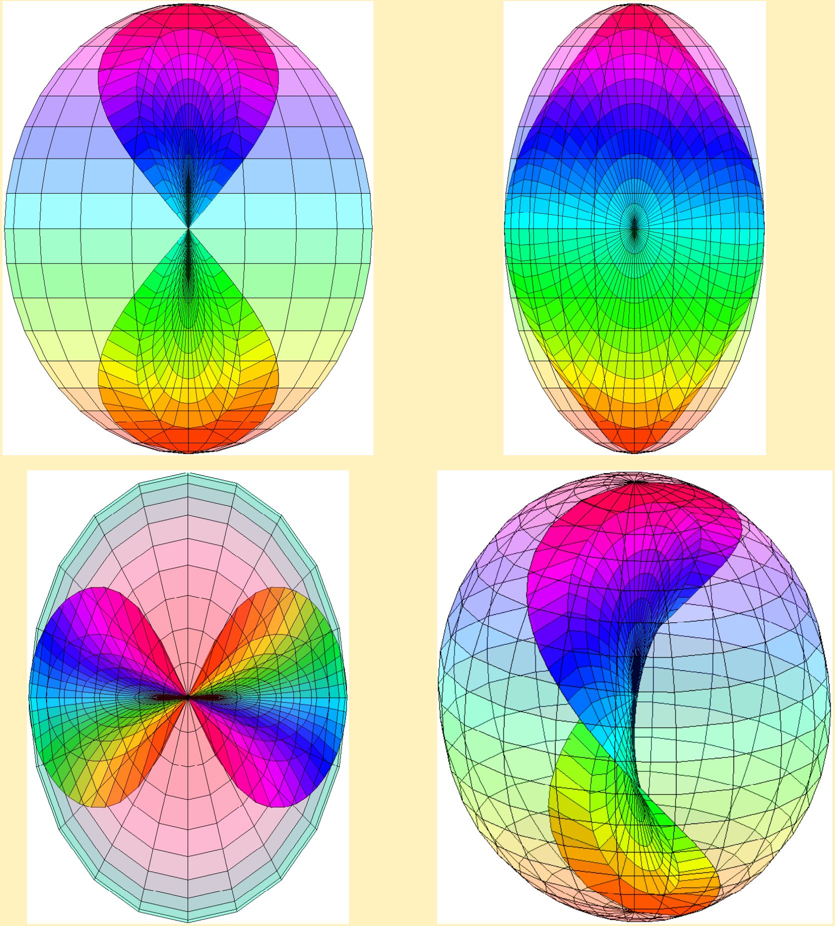
General control weighting

- We could not find an explicit expression for the value function V when R is not a multiple of the identity matrix.
- We have solved the optimization problem numerically in order to explore the relationship between the weighting matrix R and value function V.
- We can restrict our attention *without loss of generality* to diagonal positive definite weighting matrix R.
- For the special case $R = rI$, we concluded that the set of non-differentiable points for the value function is Π. According to numerical evidence, we *claim* that this is also true for an arbitrary positive definite diagonal weighting matrix.
- The infinite time horizon optimal control problem satisfies

$$V(g(0)) := \min_{\xi(\cdot)} \frac{1}{2} \int_0^\infty l(g(\tau), \xi(\tau))\, d\tau = \\ = \min_{\xi(\cdot)} \left\{ \frac{1}{2} \int_0^T l(g(\tau), \xi(\tau))\, d\tau + V(g(T)) \right\} \tag{3}$$

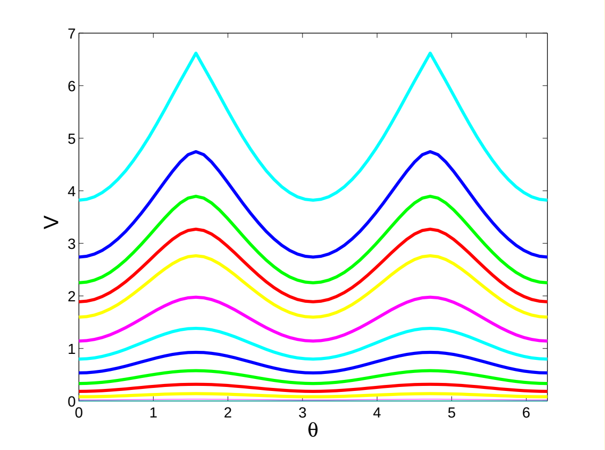
where $\dot{g}(t) = \hat{\xi}(t) g(t)$, $g(0) = g_0$. V is a suitable approximation of the value function around the origin (Jadbabaie *et al.*, 2001).

General control weighting (cont'd)



- Image of the x-z disk of radius one through the mapping $\mu_s(\cdot) : \mathsf{SO}(3)\backslash\Pi \to \mathfrak{so}^*(3)$
- $\mathsf{SO}(3)\backslash\Pi \approx B^{\mathbb{R}^3}_{[0,1)}$
- $\mathfrak{so}^*(3) \approx \mathbb{R}^3$
- $\|\mu\|^2_R \leq 4$
- $\xi^*(t) = -R^{-1}\mu(t)$
- $R = \text{diag}(1, 2, 3)$

Kinks along the ridge



- A very interesting phenomenon, which has not an explanation yet, has being noted when the weighting matrix R has two equal elements. A representation of the value function for the case $R = (1, 1, 3)$ is shown.
- $x_0(\rho, \theta) = [\rho \cos\theta \, 0 \, \rho \sin\theta]^T$
- Different value of the radial distance $\rho \in 0.999 \times \{10^{-3} \, .1 \, .2 \, .3 \, .4 \, .5 \, .6 \, .7 \, .8 \, .85 \, .9 \, .95 \, 1\}$ and for $\theta \in [0, 2\pi]$.
- The value function appear to have a ridge not only as we approach Π but a kink also appears as we consider the value of V on a series of concentric spheres whose radius (ρ) tends to one.

Conclusion and future work

- We have presented an optimal stabilizing controller for the driftless dynamics $\dot{g}(t) = \hat{\xi}(t)g(t)$, $g(0) = g_0$, showing that a closed form solution exist for the special case $Q = I$ and $R = rI$
- We have studied the nature of the optimal solution by means of numerical optimization for a general weight R and $Q = I$.
- We are interested to further investigate the optimal solution for an arbitrary weighting matrix R and introduce a general weighting matrix Q for the rotational matrix g.
- The numerical exploration of the solutions for this problem using the weighted Frobenius norm $\|e - g\|_Q^2$ has shown to be *much* more efficient using the Lie group projection operator (than the standard flat space approach).
- We are investigating the convergence rate of the standard projection operator approach (based on quaternion parametrization) against the Lie group projection operator approach

Saccon, A., Hauser, J., and Aguiar, P., *Optimal Control on Non-Compact Lie Groups: A Projection Operator Approach*, accepted at 49th IEEE Conference on Decision & Control, 2010

Saccon, A., Hauser, J., and Aguiar, P. *Optimal Control on Non-Compact Lie Groups: A Projection Operator Approach*, To be submitted to IEEE Transactions of Automatic Control, 2010

Thank you for your attention!