THE UNIVERSITY OF MELBOURNE

Cooperative Stabilisation & k-Pairs Communication Networks

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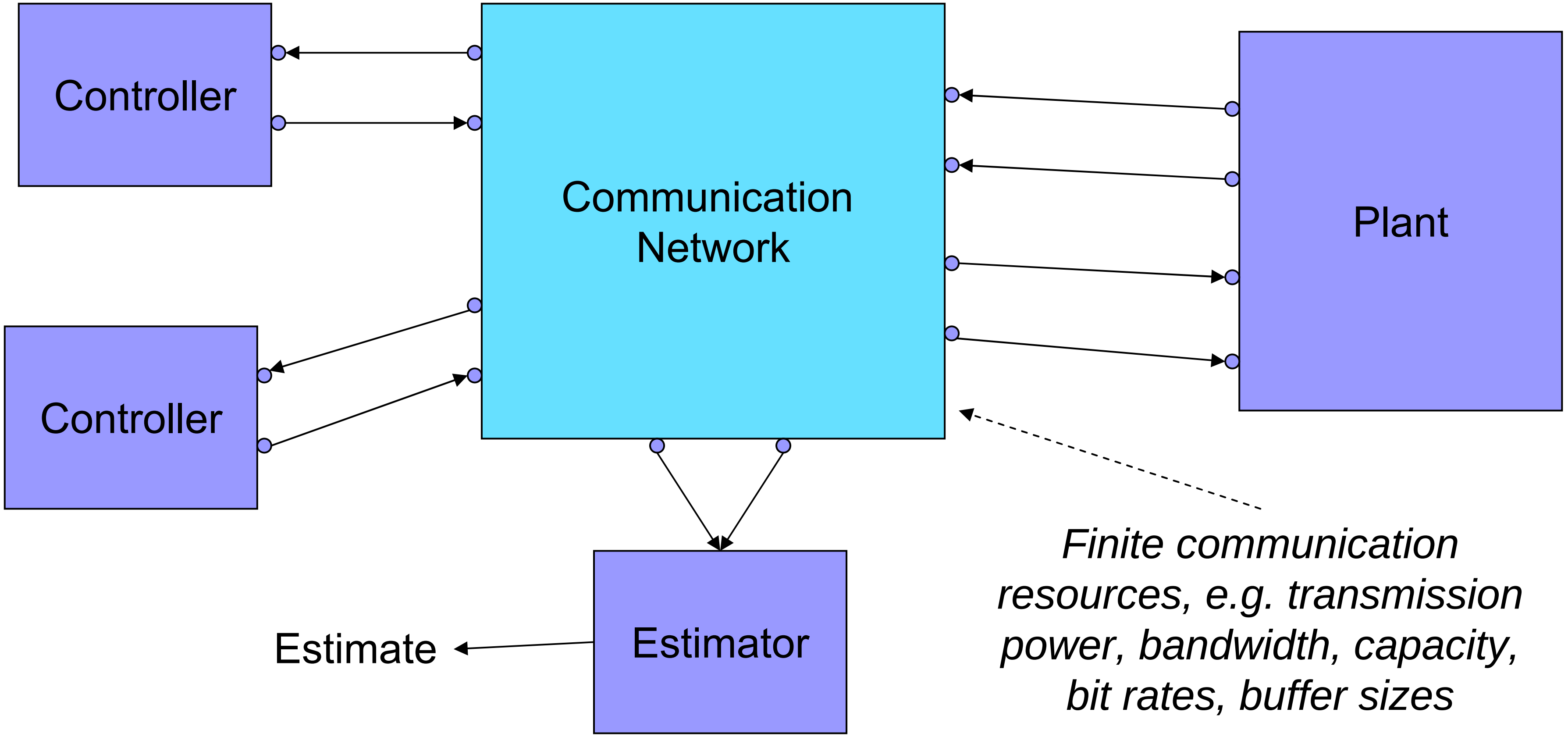
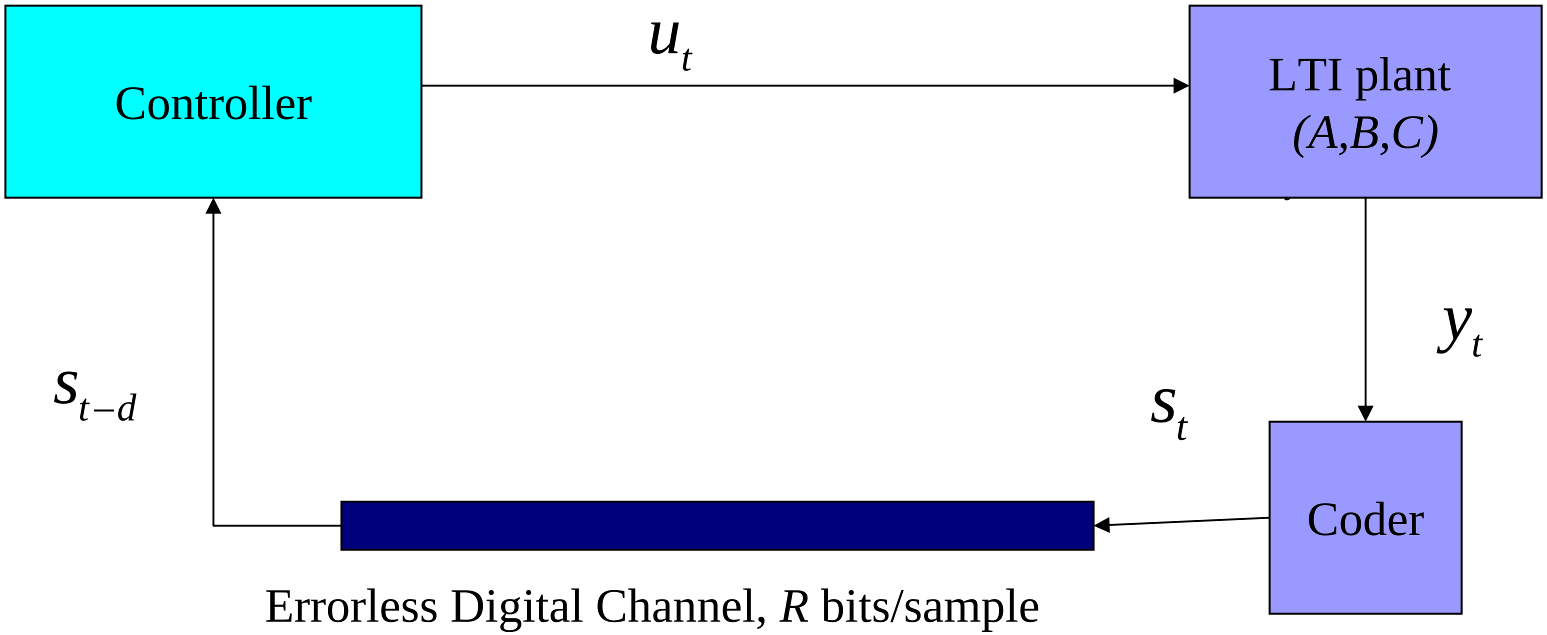
NecSys'09, Venice, Italy
25/9/09

Networked Control Systems



Single Loop



Q: Given no a priori *coder-controller constraints but causality, is stability possible given R?*

The Data Rate Theorem

A coder-controller that stabilises the plant exists *iff*

$$R > H := \sum_{\lambda \in \sigma(A), |\lambda| \geq 1} \log_2 |\lambda| \quad \text{(bits/sample)}$$

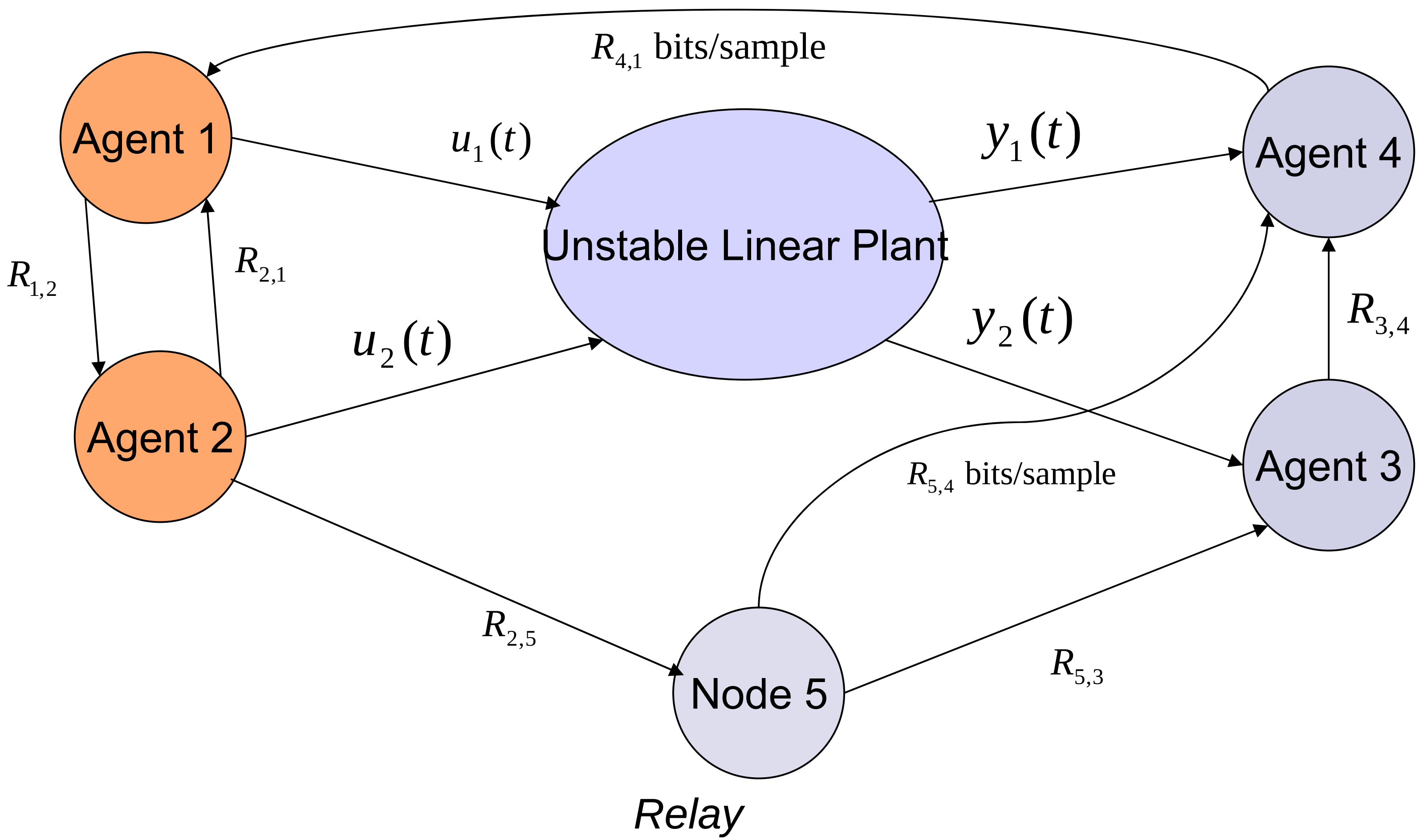
Holds for different formulations & stability definitions!

Deterministic plant, bounded states: *Baillieul (Proc. Stoch. The. Contr. Workshop'01), Hespanha et. al, MTNS '02, Tatikonda & Mitter (TAC '04)*

Noiseless plant w. unbounded random initial state, moment stability: *N. & Evans (Automatica '03)*

Unbounded noise, mean sq. stability : *Nair & Evans (SIAM J. Cont. Opt. '04)*

Cooperative Networked Control



Cooperative Networked Control: Plant Formulation

$$X(t+1) = AX(t) + \sum_{i=1}^{N} B_i U_i(t) + V(t) \in \mathbf{R}^n,$$

$$Y_i(t) = C_i X(t) + W_i(t) \in \mathbf{R}^{q_i}, \qquad i = 1, \ldots, N.$$

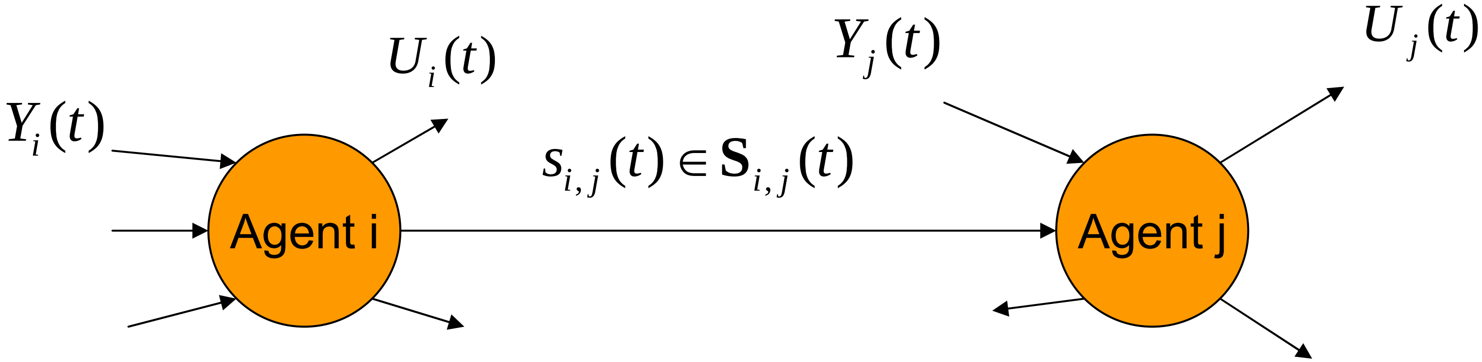
$\mathbf{A1}$: $X(0), V(s)$ & $W_i(t)$ $(s, t \geq 0,\ 1 \leq i \leq N)$ are mutually independent.

$\mathbf{A2}$: $X(0)$, $\{V(t)\}$ and $\{W_i(t)\}$ are mean-square bounded, with $X(0)$ having an absolutely continuous distribution.

$\mathbf{A3}$: The plant is controllable by all inputs together & observable from all outputs together.

NB: Some agents may have B_i and/or C_i =0.

Channel Formulation



$$(\textit{Average})\ \textit{Channel Data Rate}\ \ r_{i,j} := \underline{\lim}_{t\to\infty} \frac{1}{t}\sum_{k=0}^{t-1} \log_2 \left|\mathbf{S}_{i,j}(k)\right| \ \ \text{(bits/sample)}$$

No direct communication from $(i \rightarrow j) \Rightarrow$ set $\left|\mathbf{S}_{i,j}(k)\right| = 1 \Rightarrow r_{i,j} = 0.$

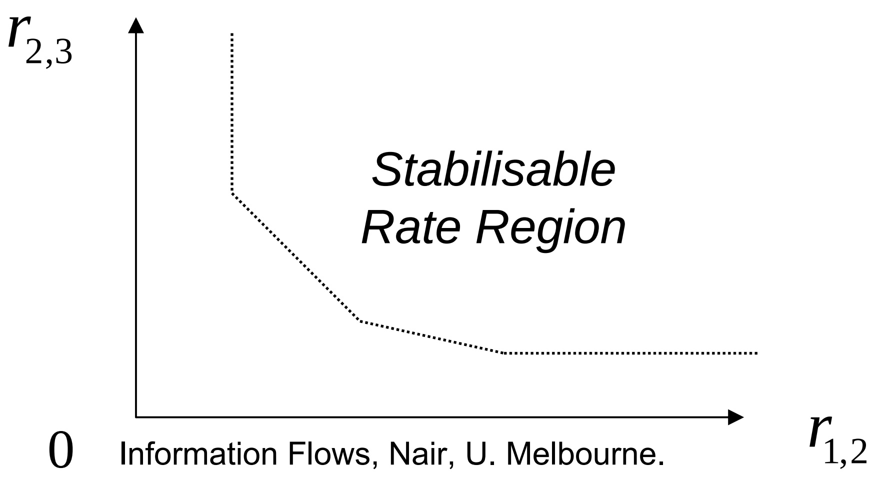
Perfect communication from $(i \rightarrow j) \Rightarrow$ set $\left|\mathbf{S}_{i,j}(k)\right| = \infty \Rightarrow r_{i,j} = \infty.$

Achievable Rate Region

Let $r := \left(r_{i,j}\right)_{1 \le i \ne j \le N} \in \mathbf{R}_{\ge 0}^{N(N-1)}$

Question : What is the region of rate tuples r for which there exists a cooperative scheme that mean-square stabilises the plant state, under no constraints but causality on the agents?

E.g.



Previous Literature

- Classical decentralised LTI control:

$$u_j(t) = K_j(s) y_j(t).$$

- *Wang & Davison, IEEE TAC '73* : Stabilisability = No unstable decentralised fixed modes.
- *Corfmat & Morse, Automatica'76:* Spectrum assignability = Completeness & strong connectedness.
- *Kobayashi et. al., TAC'78:* Decentralised controllability

  (2 controllers)
- *Anderson & Moore, IEEE TAC `81 –* strong connectedness suffices for stabilisability with LTV controllers.

Bit-rate-constrained formulations:

- *Noiseless plant, multiple sensors & one controller*

  -Sufficient condition (Tatikonda & Mitter, Allerton `00)

  -Necessary & sufficient condition (Matveev & Savkin, SIAM J. Cont. Opt. `06)
- *Noiseless plant, multiple sensors & controllers*

  -Separate necessary & sufficient conditions(N. et. al., CDC`04)

  - Equivalence to multiterminal source coding (Matveev & Savkin, Birkhauser `09)

  - Strong connectedness (Yuksel & Basar TAC`06)

Marginal Information Demand

Fact : By abs. cont., $\exists$ a sufficiently small hypercube $\mathbf{H}$ aligned with $X-$coordinate axes

$$\text{s.t. } \inf_{x\in\mathbf{H}} f_{X(0)}(x) > 0.$$

$\Rightarrow$ Mean-square stability maintained if $X(0) \sim \mathrm{U}(\mathbf{H}) = \mathrm{U}(\mathbf{H}_1)\times\cdots\times\mathrm{U}(\mathbf{H}_n)$.

$\Rightarrow$ Assume w.l.o.g. that $X_1(0),\ldots,X_n(0)$ are mutually independent.

W.l.o.g., also assume plant is in real Jordan form.

Lemma : $$\mathrm{I}_\infty\left\{X_h(0);\Psi_{\mathbf{IN}(\mathrm{D}_h)}\right\} := \overline{\lim_{t\to\infty}}\frac{1}{t+1}\mathrm{I}\left\{X_h(0);\Psi_{\mathbf{IN}(\mathrm{D}_h)}(0),\ldots,\Psi_{\mathbf{IN}(\mathrm{D}_h)}(t)\right\} \geq \log_2^+|\lambda_h|,$$

where λ_h = plant eigenvalue governing X_h

$\Psi_{\mathbf{IN}(\mathrm{D}_h)} :=$ tuple of all signals available to agents that can directly affect X_h,

$\mathrm{I}\{X;Y\} :=$ mutual information (in bits) between rv's X & Y.

Marginal information about initial h-th mode must be received at this combined rate by all agents that can affect it.

Communication Graph

Seek conditions allowing this demand to be met by inducing a capacitated signalling digraph

$$G := \left(\mathbf{V}, \mathbf{E}, c, \Psi\right).$$

$$\mathbf{V} := \left\{\text{Agents } 1, \ldots, N\right\} \bigcup \left\{\textit{Sources } \mathrm{S}_1, \ldots, \mathrm{S}_n\right\} \bigcup \left\{\textit{Destinations } \mathrm{D}_1, \ldots, \mathrm{D}_n\right\}.$$

$\mathbf{E} :=$ set of directed edges (arcs) between certain ordered vertex pairs.

(An arc representing a system output/exogeneous input
is allowed not to originate/terminate in a vertex.)

$c := (c_e)_{e \in \mathbf{E}}$, tuple of arc capacities ≥ 0.

$\Psi := (\Psi_e)_{e \in \mathbf{E}}$, tuple of signals carried on each arc s.t.

$$\forall v \in \mathbf{V}, \quad \Psi_{\mathbf{OUT}(v)} = \text{causal functional of } \Psi_{\mathbf{IN}(v)}$$

$$\& \ \ \forall e \in \mathbf{E}, \ \ \mathrm{I}_\infty \left\{\Psi_e ; \Psi_{\mathbf{IN}(G)}\right\} \leq c_e .$$

Arc e	Signal $\Psi_e(t)$	Capacity c_e
$(\mathrm{A}_i \to \mathrm{A}_j)$	$S_{i,j}(t)$	$r_{i,j}$
$(\mathrm{D}_h \to)$	$\hat{X}_h(0\|t)$	∞
$(\to \mathrm{S}_h)$	$X_h(0)$	∞
$(\mathrm{A}_i \to \mathrm{D}_h)$	??	??
$(\mathrm{S}_h \to \mathrm{A}_i)$	??	??
$(\mathrm{D}_h \to \mathrm{S}_h)$	??	??

A ‘Relaxation’

Agent-i receives noisy linear combination $Y_i(t) = C_i X(t) + W_i(t)$ of modes.

Similarly, each $X_h(t)$ "sees" a noisy linear combination of agents' control actions.

$\Rightarrow$ *MIMO* communication channels.

Capacity region is major open problem in IT, even for 2x2...

(see e.g. work of D. Tse. et al & T.S. Han)

$\Rightarrow$ *Endow* any agent directly affected by a mode with *perfect* knowledge of it

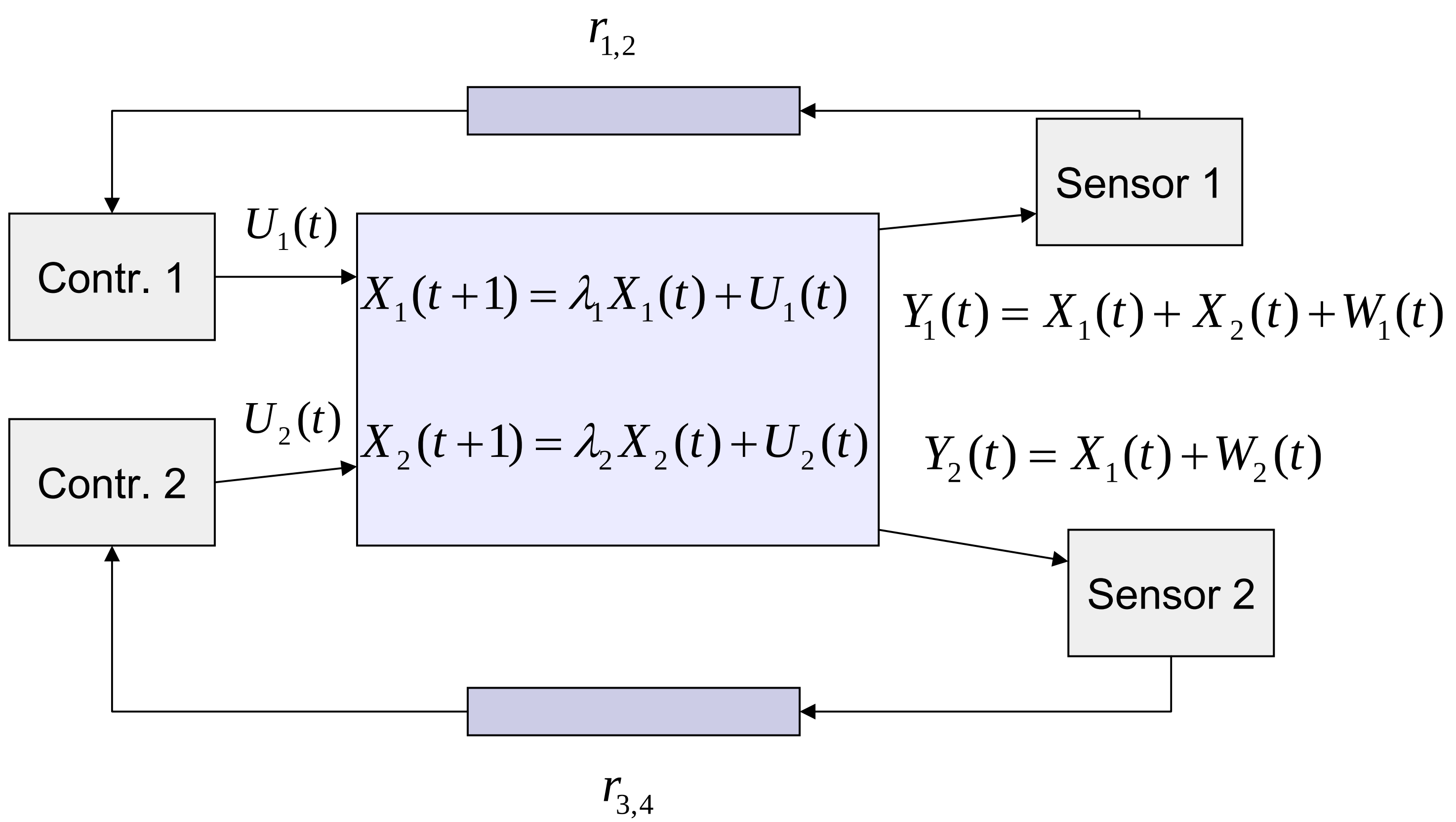
& any agent which directly affects a mode, *perfect* transmission to it.

$$\text{Let } d_{h,i} := \begin{cases} 1 & \text{if } h\text{th row of input matrix } B_i \text{ is } \neq 0 \\ 0 & \text{otherwise} \end{cases},$$

$$e_{i,h} := \begin{cases} 1 & \text{if } h\text{th column of output matrix } C_i \text{ is } \neq 0 \\ 0 & \text{otherwise} \end{cases}.$$

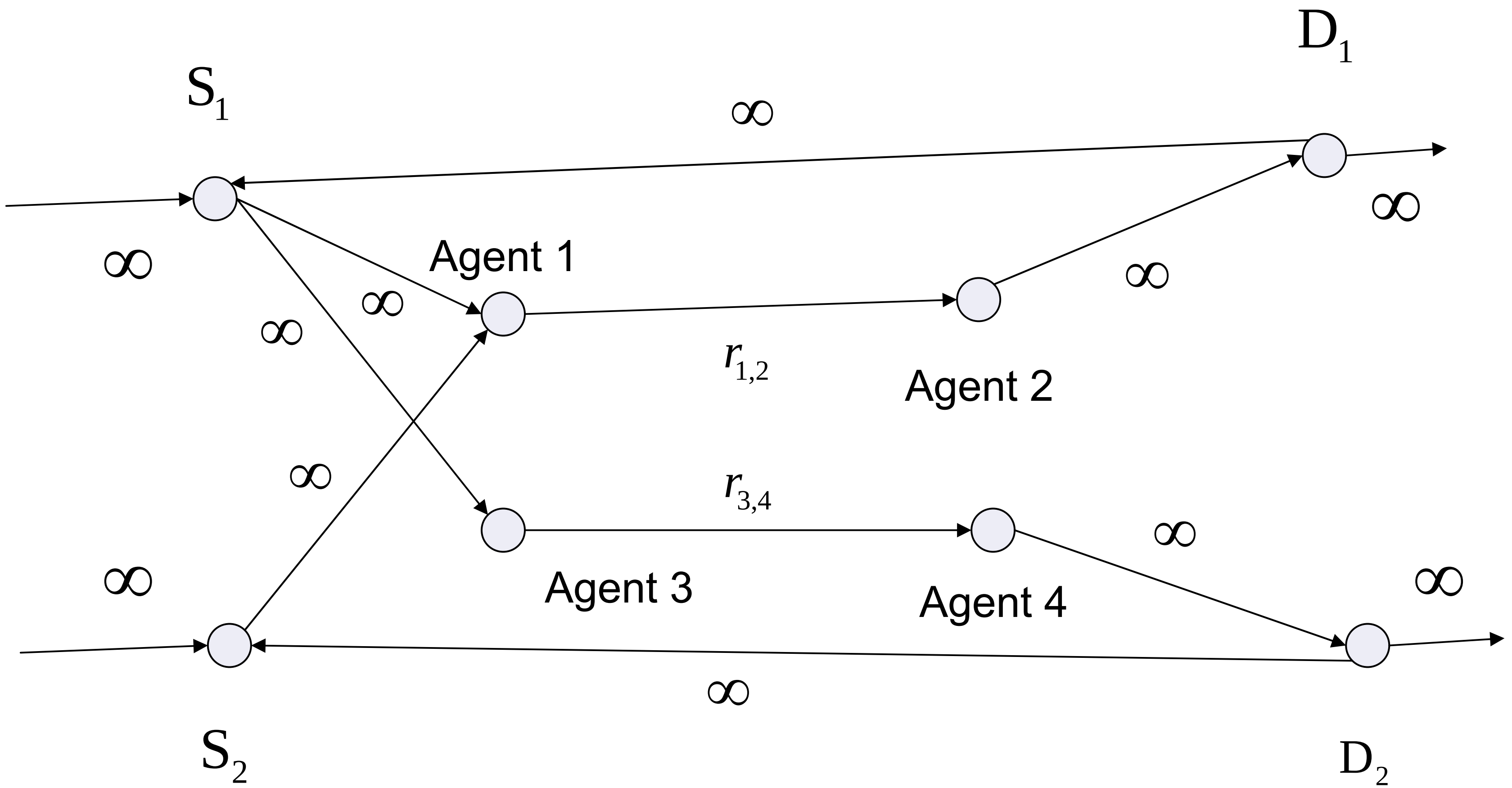
Arc e	Signal $\Psi_e(t)$	Capacity c_e
$(\mathrm{A}_i \to \mathrm{A}_j)$	$S_{i,j}(t)$	$r_{i,j}$
$(\mathrm{D}_h \to)$	$\hat{X}_h(0\|t)$	∞
$(\to \mathrm{S}_h)$	$(X_h(0),\ V_h(t-1))$	∞
$(\mathrm{A}_i \to \mathrm{D}_h)$, iff $e_{i,h}=1$	$U_i(t)$	∞
$(\mathrm{S}_h \to \mathrm{A}_i)$, iff $d_{h,i}=1$	$X_h(t)$	∞
$(\mathrm{D}_h \to \mathrm{S}_h)$	$(U_i(t-1))_{i:e_{i,h}=1}$	∞

2x2 Example



$r_{1,2}$
Sensor 1
Contr. 1
$U_1(t)$
$X_1(t+1) = \lambda_1 X_1(t) + U_1(t)$
$Y_1(t) = X_1(t) + X_2(t) + W_1(t)$
$U_2(t)$
Contr. 2
$X_2(t+1) = \lambda_2 X_2(t) + U_2(t)$
$Y_2(t) = X_1(t) + W_2(t)$
Sensor 2
$r_{3,4}$

Capacitated Digraph for Example



n-Pairs Communication Network

If mean square stability achieved, then each destination D_h receives marginal information over this capacitated digraph about corresponding source at rate $\geq \log_2^+ |\lambda_h|$.

➔ *n-Pairs communication problem.* Rate region not generally known! (see e.g. N. Harvey et. al.,M. Adler et al, Kramer & Savari 2006)

Some differences from standard formulation:

- Each source is a static, continuous rv with infinite information content, not an iid discrete sequence with finite entropy rate.
- Cycles present due to feedback.

See e.g. T.S. Han '80, Ahlswede et al '00, T.S. Han '09 for solutions to certain other multiterminal network information problems.

h-Paths

Let a *h*-path := any simple path from S_h to D_h

& the *h*-bundle := set of all *h*-paths.

E.g., for the example,

$$1\text{-paths} = (S_1, A_1, A_2, D_1) \ \& \ (S_1, A_3, A_4, \boxed{D_2, S_2}, A_1, A_2, D_1).$$

$$2\text{-path} = (S_2, A_1, A_2, \boxed{D_1, S_1}, A_3, A_4, D_2)$$

Each *h*-path = loopless route by which information about $X_h(0)$ may be conveyed from S_h to D_h.

These paths may involve signalling *through* the plant

Multicommodity Fluid Flow?

Intuitively, we would *like* to believe that $X_h(0)$-information flows like an immiscible, incompressible fluid through the h-bundle from S_h to D_h.

I.e., on each h-path $\mathbf{p}$, we'd like there to $\exists$ a h-path flow $\varphi_{\mathbf{p}} \geq 0$ s.t.

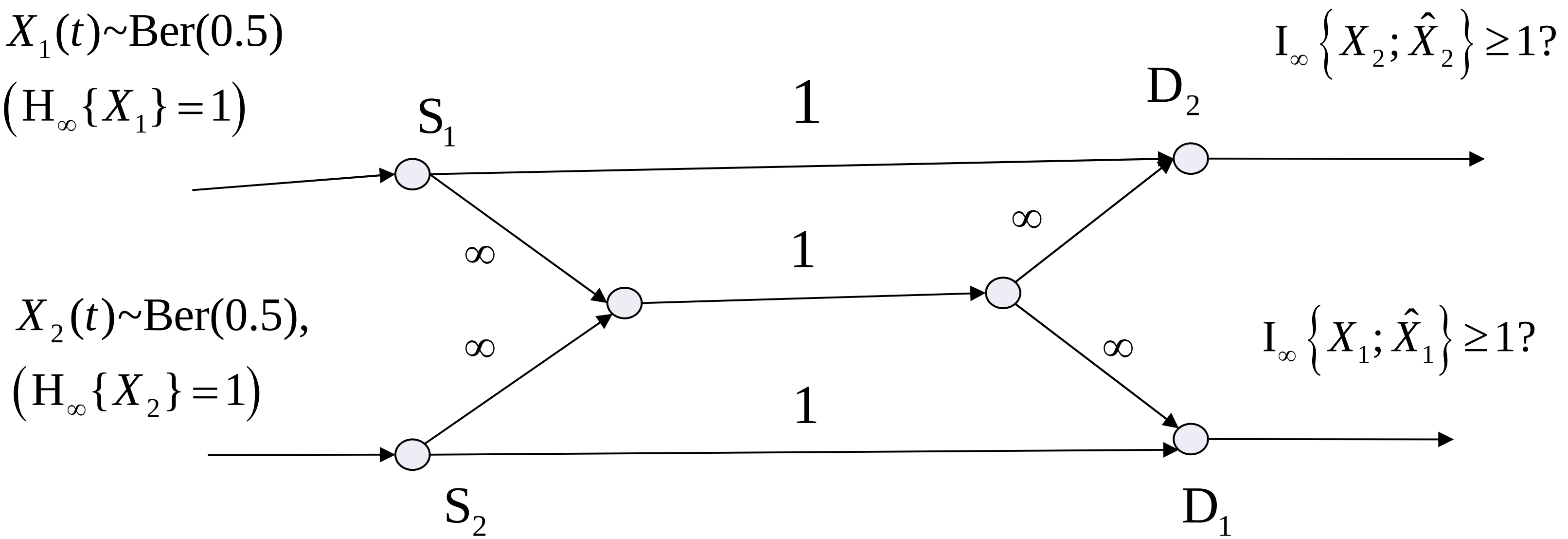
1. $\forall$ Arcs e, $\sum_{\text{all } h\text{-paths } \mathbf{p} \text{ traversing } e} \varphi_{\mathbf{p}} \leq c_e$,
2. $\forall h \in [1, \ldots, n]$, $\sum_{\mathbf{p} \in h\text{-bundle}} \varphi_{\mathbf{p}} \geq \log_2^+ |\lambda_h|$.

(Conservation is implicitly satisfied at every vertex v, since if $\varphi_{\mathbf{p}}$ enters v on h-path $\mathbf{p}$ it also leaves along the same h-path.)

Unfortunately, this is not generally possible.

Counterexample

(after Ahlswede et. al., Trans. IT 2000)



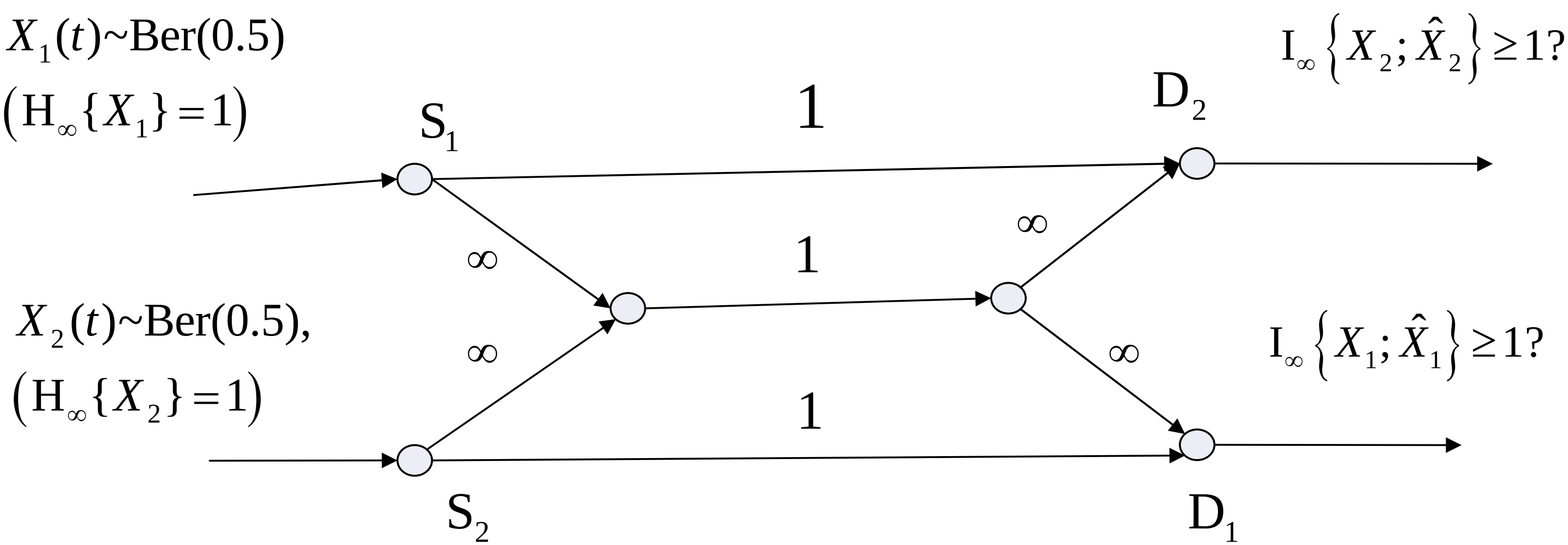
Fluid view: flow on outer channels must be 0.

Inner channel carries 0.5bits/sample from X_1 & X_2 each.

$\Rightarrow$ Seems problem is infeasible.

Counterexample

(after Ahlswede et. al., Trans. IT 2000)



Ahlswede et al: let $S_1 = X_1,\ S_2 = X_2,\ S_3 = X_1 \oplus X_2$ (mod 2 addition)

Then let $\widehat{X}_2 = S_1 \oplus S_3 = X_1 \oplus X_1 \oplus X_2 = X_2$,

$$\widehat{X}_1 = S_2 \oplus S_3 = X_2 \oplus X_1 \oplus X_2 = X_1.$$

$\Rightarrow$ Perfect reconstruction possible & problem is feasible.

Triangularity

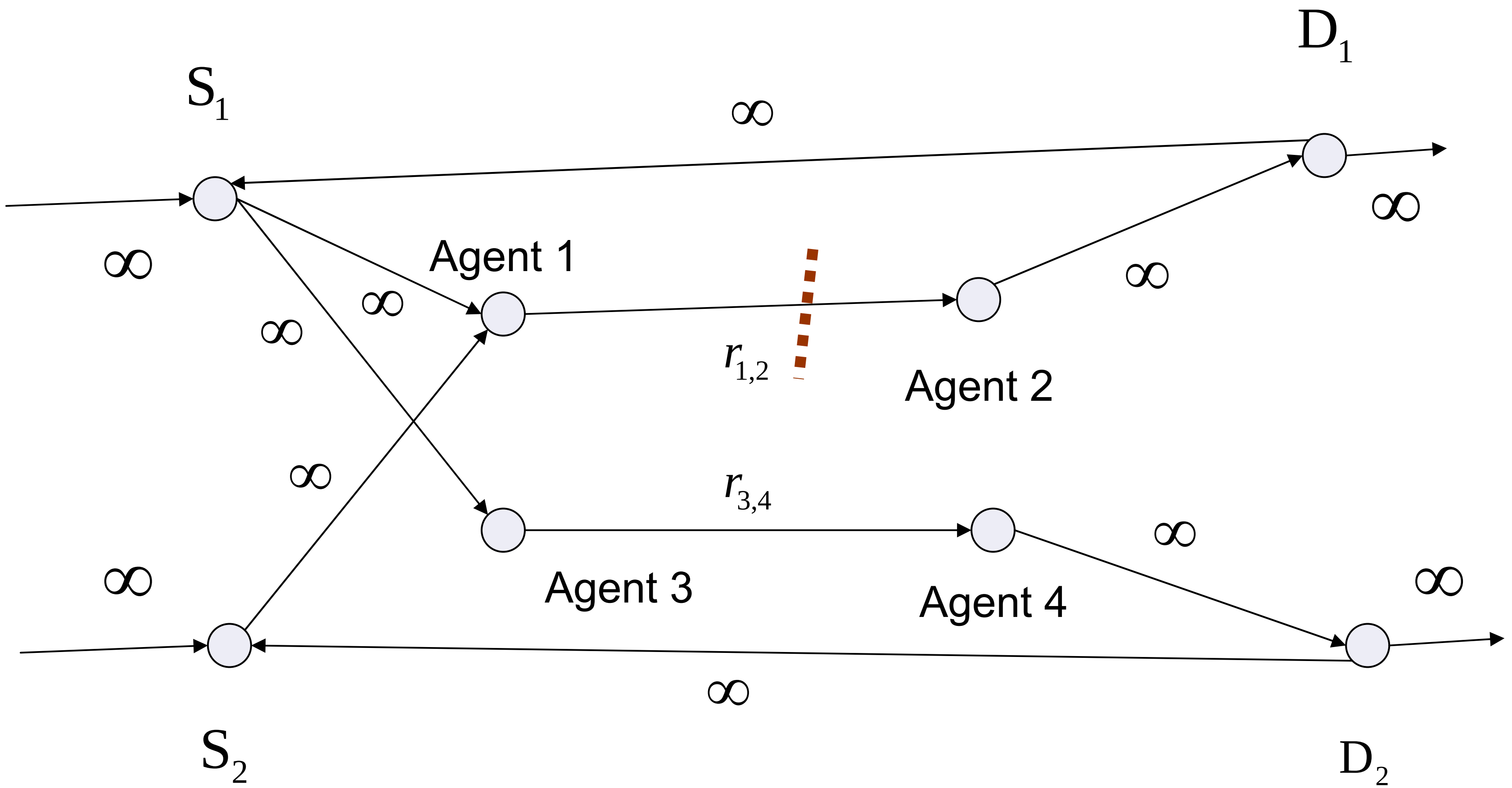
Nonetheless, being able to treat network as routing separate end-to-end streams of info. would be conceptually simple & practically useful.

$\Rightarrow Q:$ For what class of networked control systems can this always be done?

Definition: A capacitated n-pairs digraph $(\mathbf{V},\mathbf{E},c)$ is *triangular* if $\exists$ an indexing $h=1,\ldots,n$ of the source-destination pairs s.t.

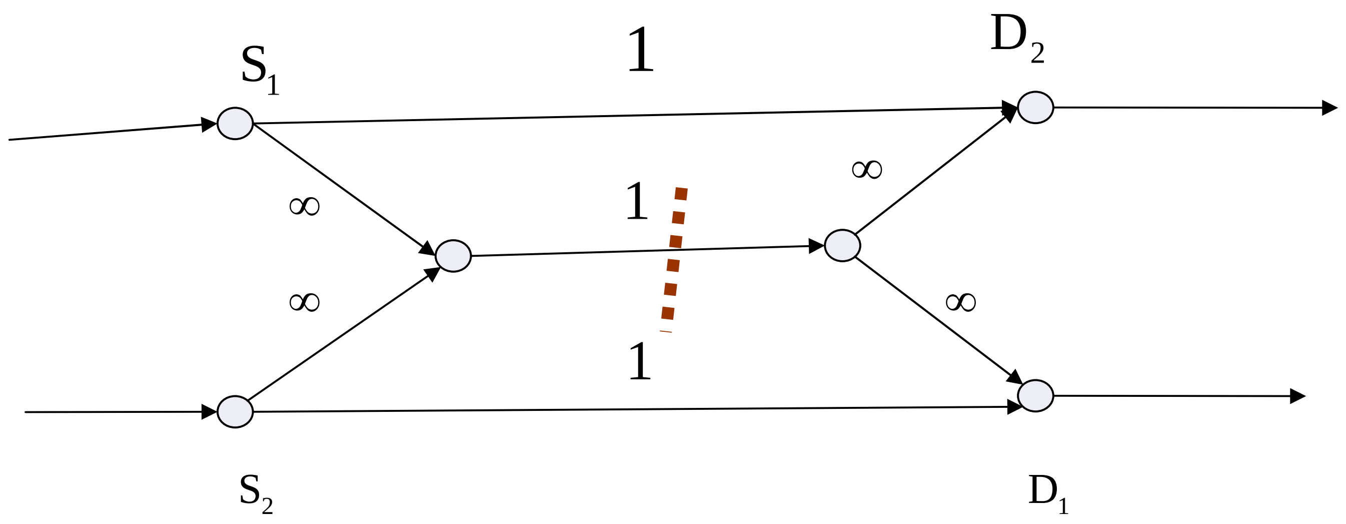
$i)$ Each source S_h has a h-path &
$ii)$ Each finite-capacity minimal cut of a h-bundle also cuts off D_h from $\mathrm{S}_1,\ldots,\mathrm{S}_{h-1}$, $\forall h\in[2,\ldots n]$.

Example Triangular



The only (minimal) finite-capacity cut of the 1-bundle also cuts source-2 from dest.-1.

Ahlswede Counterexample Not Triangular!



Tight Characterisation of Rate Region

Under Assumptions 1-3 on the LTI plant, if mean square stability is achieved & the induced capacitated digraph is triangular, then the digraph must support a multicommodity flow. I.e., on each h-path $\mathbf{p}$, $\exists\ \varphi_{\mathbf{p}} \geq 0$ s.t.

$$\forall \text{ Arcs } e, \sum_{\text{all } h\text{-paths } \mathbf{p} \text{ traversing } e} \varphi_{\mathbf{p}} \leq c_e,$$

$$\forall h \in [1, \ldots, n], \sum_{\mathbf{p} \in h\text{-bundle}} \varphi_{\mathbf{p}} \geq \log_2^+ |\lambda_h|. \quad (*)$$

If there does exist such a multicommodity flow $\left(\text{with strict inequality in } (*)\right)$, the plant has distinct eigenvalues and the noise & initial state are bounded, then a cooperative networked coding & control scheme can be constructed to achieve (mean square) bounded states.

Stabilisability Criterion for Example

For stability to be possible, there must exist $\rho_{1,1}, \rho_{1,2}, \rho_{2,1} \geq 0$ s.t.

$$\rho_{1,1} + \rho_{1,2} + \rho_{2,1} \leq R_1, \quad \rho_{1,2} + \rho_{2,1} \leq R_2$$

$$\rho_{1,1} + \rho_{1,2} \geq \log_2 |\lambda_1|, \quad \rho_{2,1} \geq \log_2 |\lambda_2|.$$

$$\Leftrightarrow \; R_1 \geq \log_2 |\lambda_1| + \log_2 |\lambda_2|, \; R_2 \geq \log_2 |\lambda_2|$$

$$\left(\begin{array}{c} \Rightarrow R_1 + R_2 \geq \log_2 |\lambda_1| + 2\log_2 |\lambda_2|. \\ \text{C.f. centralised condition, } R_1 + R_2 \geq \log_2 |\lambda_1| + \log_2 |\lambda_2|. \end{array} \right)$$

Centralised Result Recovered

Every mode x_h has exactly one irreducible cycle, passing over the single rate R channel

$\Rightarrow$ Stabilisability criterion reduces to the existence of $\rho_h \geq 0$ for each $h-$th mode, s.t.

$$R \geq \sum_h \rho_h, \qquad \rho_h \geq \log_2 |\eta_h|, \quad \forall h.$$

$\equiv$ Known criterion $R \geq \sum_h \log_2 |\eta_h|.$

Comparison with Decentralised LTV Control

In classical decentralised control, no channels between agents.

By joint controllability & observability,
every mode x_h affects some agent and is affected by another, possibly different agent.
Combined with strong connectedness $\Rightarrow \forall\ h, \exists$ a h-path.

Arcs all have ∞ capacity $\Rightarrow$ Can always choose $\varphi_{\mathbf{p}}$ sufficiently large on each h-path $\mathbf{p}$ to meet demand.

$\Rightarrow$ Decentralised stability is possible, agreeing with Anderson & Moore '81.

Conclusions & Future Work

- Information is not generally a flow!
- However, certain nontrivial classes of networked control systems are stabilisable iff they support multicommodity-like information flows.

- Noisy channels?
- Unbounded plant noise?
- Generalisation to coordination problems (i.e. stabilisation to a subspace)?