Improving the stability conditions of TS fuzzy systems with fuzzy Lyapunov functions

Flávio A. Faria ∗ Geraldo N. Silva ∗ Vilma A. Oliveira ∗∗ Rodrigo Cardim ∗∗∗

* Departamento de Ciência da Computação e Estatística, Instituto de Biociências Letras e Ciências Exatas, UNESP, 15054-000, São José do Rio Preto, SP, Brazil (e-mail: flaviof15@yahoo.com.br; gsilva@ibilce.unesp.br).

** Departamento de Engenharia Elétrica, Universidade de São Paulo, 13560-970, São Carlos, SP, Brazil (e-mail: vilma@sc.usp.br)

*** Departamento de Engenharia Elétrica, UNESP, Campus de Ilha Solteira, 15385-000, SP, Brazil (e-mail: rcardim@dee.feis.unesp.br)

Abstract: In this paper, the fuzzy Lyapunov function approach is considered for stabilizing continuous-time Takagi-Sugeno fuzzy systems. Previous linear matrix inequality (LMI) stability conditions are relaxed by exploring further the properties of the time derivatives of premise membership functions and by introducing a slack LMI variable into the problem formulation. The stability results are thus used in the state feedback design which is also solved in terms of LMIs. Numerical examples illustrate the efficiency of the new stabilizing conditions presented.

Keywords: TS fuzzy system; Fuzzy Lyapunov functions; Stability analysis; State feedback design; Linear matrix inequalities.

1. INTRODUCTION

During the last years, the Takagi-Sugeno (TS) fuzzy modeling has been considered in the development of analysis and control techniques for nonlinear systems, as they can be considered universal approximators for some classes of nonlinear systems (Taniguchi et al., 2001). The main idea of the TS fuzzy modeling is to obtain linear time-invariant models “close” to the nonlinear system in some regions of the state-space and then combine these linear models using nonlinear fuzzy membership functions (Takagi and Sugeno, 1985; Sugeno and Kang, 1988). The attractiveness of the TS fuzzy modeling is that the stability analysis and design of controllers for nonlinear systems can be formulated in the framework of LMIs, which can be efficiently solved by convex programming techniques (Boyd et al., 1994).

Stability and stabilization of the TS fuzzy models are usually investigated via the direct Lyapunov method, and most of the works use common quadratic Lyapunov function (CQLF) to guarantee stability. Currently, much effort has been devoted to obtaining less conservative conditions with this approach (Tanaka et al., 1998; Teixeira et al., 2003; Fang et al., 2006). However, it is well-known that the CQLF leads to conservative results and that in some cases it is not even possible to find a CQLF (Johansson et al., 1999). Therefore, several approaches have been developed to overcome these drawbacks, among which, piecewise linear Lyapunov functions (PLFs) and fuzzy Lyapunov functions (FLFs) have attracted a lot of attention.

PLFs have been investigated as a natural option when the TS modeling does not have all linear models activated at once (Johansson et al., 1999; Feng, 2003; Arrifano et al., 2006; Bernal et al., 2009). This approach decreases the conservatism of the stability analysis by partitioning the system state space region into a number of polyhedral regions.

The fuzzy Lyapunov function approach consists of finding a symmetric positive matrix for each linear time-invariant model, and then construct a multiple Lyapunov function using the same membership function of the TS fuzzy system (Tanaka et al., 2003; B.-J. Rhee, 2006; Mozelli et al., 2009a,b). Usually, the stability conditions obtained with PLF depend explicitly on the time-derivative of membership functions. To convert these conditions into LMIs, upper bounds for the time-derivatives of the membership functions are usually considered. To decrease the conservatism of the LMI’s solution, Tanaka et al. (2003) explored relevant properties of the membership functions. To improve these results, the addition of a slack matrix variable into the LMIs conditions to decouple the Lyapunov matrix providing new degrees of freedom for solving the LMI-based problem has been studied recently (Mozelli et al., 2009a,b). The method proposed in Mozelli et al. (2009a) allows the state feedback design be accomplished using LMIs, differently from the methods presented in Tanaka et al. (2003); B.-J. Rhee (2006) which used bilinear matrix inequalities (BMIs).
In this work, new stability conditions are obtained by further relaxing the LMIs conditions presented in Tanaka et al. (2003) and considering the addition of an slack variable proposed in Mozelli et al. (2009b). The new stability conditions allow more degrees of freedom into the stability analysis and control design for TS fuzzy systems than the conditions given in Tanaka et al. (2003); B.-J. Rhee (2006); Mozelli et al. (2009a,b), which are illustrated by numerical examples. Furthermore, a state feedback design using the same scheme is solved with LMIs.

Throughout this paper \( M > 0 \) (\( M \succeq 0 \)) means that the matrix \( M \) is symmetric and positive definite (semidefinite) matrix. Similarly, \( M < 0 \) (\( M \preceq 0 \)) means that the matrix \( M \) is symmetric and negative definite (semidefinite) matrix. The symbol \( \prec \) within a matrix represents the symmetric terms of the matrix.

2. TAKAGI-SUGENO FUZZY MODEL

The model proposed by Takagi and Sugeno (1985) is described by fuzzy rules, which represent local linear relations of a nonlinear system. The global TS fuzzy model is given by:

\[
\dot{x}(t) = \sum_{k=1}^{r} h_k(z(t))A_k x(t) \tag{1}
\]

where \( z(t) \in \mathbb{R}^p \) is the premise vector, \( x(t) \in \mathbb{R}^n \) is the state vector, \( A_k \in \mathbb{R}^{n \times n} \) are matrices of the local models, \( r \) is the number of model rules and \( h_k(z(t)) \) is the normalized weight for each local system \( \dot{x}(t) = A_k x(t) \) which satisfies the following properties:

\[
\forall k \in \mathcal{R}, \quad h_k(z(t)) \geq 0 \quad \text{and} \quad \sum_{k=1}^{r} h_k(z(t)) = 1 \tag{2}
\]

with \( \mathcal{R} = \{1, 2, \ldots, r\} \). For simplicity, in what follows, \( h_k(z(t)) \) will be denoted by \( h_k \).

3. RECENT STABILITY CONDITIONS WITH FUZZY LYAPUNOV FUNCTIONS

The first result on the stability of TS fuzzy systems with FLF was presented in Tanaka et al. (2003) and is repeated here for easy reference.

**Lemma 1.** Assume \( |h_\rho| \leq \phi_\rho \) for known positive real numbers \( \phi_\rho \) and \( \rho \in \mathcal{R} \). The TS fuzzy system (1) is stable if there exist symmetric matrices \( P_\rho \in \mathbb{R}^{n \times n} \), satisfying the following LMIs:

\[
P_\rho > 0, \quad \forall \rho \in \mathcal{R} \tag{3}
\]

\[
P_\rho - P_\rho' \succeq 0, \quad \forall \rho \in \mathcal{R} - \{r\} \tag{4}
\]

\[
P_\phi + \frac{1}{2} (P_k A_k + A_k' P_k + P_k A_k + A_k' P_k) < 0, \quad k \leq \ell \tag{5}
\]

where \( k, \ell = 1, \cdots, r \) and \( P_\phi = \sum_{\rho=1}^{r-1} \phi_\rho (P_\rho - P_\rho') \).

**Proof:** See Theorem 2 from (Tanaka et al., 2003).

Lemma 1 leads to considerable reduction in conservativeness as the LMI conditions are not formulated in terms of a CQLF. Recently, a less conservative result was proposed in Mozelli et al. (2009b). The result is obtained by the addition of an extra LMI variable as follows.

**Lemma 2.** Assume \( |h_\rho| \leq \phi_\rho \) for known positive real numbers \( \phi_\rho \) and \( \rho \in \mathcal{R} \). The TS fuzzy system (1) is stable if there exist symmetric matrices \( X \in \mathbb{R}^{n \times n} \) and \( P_\rho \in \mathbb{R}^{n \times n} \), satisfying the following LMIs:

\[
P_\rho > 0, \quad \rho \in \mathcal{R} \tag{6}
\]

\[
P_\rho + X \succeq 0, \quad \rho \in \mathcal{R} \tag{7}
\]

\[
P_\phi + \frac{1}{2} (P_k A_k + A_k' P_k + P_k A_k + A_k' P_k) < 0, \quad k \leq \ell \tag{8}
\]

where \( k, \ell = 1, \cdots, r \) and \( P_\phi = \sum_{\rho=1}^{r} \phi_\rho (P_\rho + X) \).

**Proof:** See Theorem 6 from (Mozelli et al., 2009b).

An example illustrating the superiority of Lemma 2 over Lemma 1 is presented next.

**Example 1**

Consider the same numerical TS fuzzy system given in (Mozelli et al., 2009b):

\[
A_1 = \begin{bmatrix} -5 & -4 \\ -1 & a \end{bmatrix}, \quad A_2 = \begin{bmatrix} -4 & -4 \\ 3b - 2 & 3a - 4 \end{bmatrix},
A_3 = \begin{bmatrix} -3 & -4 \\ 2b - 3 & 2a - 6 \end{bmatrix}, \quad A_4 = \begin{bmatrix} -2 & -4 \\ b - 2 \end{bmatrix}
\]

and

\[
h_1 = \alpha_1(x_1)^2, \quad h_2 = \alpha_1(x_1) \beta_2(x_2),
\]

\[
h_3 = \beta_1(x_1)^2, \quad h_4 = \beta_1(x_1) \beta_2(x_2)
\]

with

\[
\alpha_i(x_i) = \begin{cases} 
1 - \sin(x_i), & \text{for } |x_i| \leq \frac{\pi}{2}, \\
0, & \text{for } x_i > \frac{\pi}{2}, \\
1, & \text{for } x_i < -\frac{\pi}{2}
\end{cases}
\]

\[
\beta_i(x_i) = 1 - \alpha_i(x_i)
\]

Using MATLAB toolboxes YALMIP (Löfberg, 2004) and SeDuMi (Sturm, 1999), the stability of the system above was verified with Lemmas 1 and 2 for \( a \in [-10, -1] \), \( b \in [0, 300] \) and \( \phi_\rho = 0.85, \quad \forall k \in \mathcal{R} \). Figure 1 shows that Lemma 2 yields a larger stable region than Lemma 1.
In LMI (4) of Lemma 1, the index \( i \) of the second \( P \) is fixed at \( i = r \) and this can lead to conservativeness. As a matter of fact, by allowing the search in \( R \) for an index, the efficiency of Lemma 1 can be improved. Following this line, an extension of Lemma 1 is established in the next section.

4. NEW LESS CONSERVATIVE STABILITY CONDITIONS

In this section, less conservative stability conditions than presented in Lemmas 1 and 2 for stabilizing TS fuzzy systems (1) are given. To obtain the main results the following proposition is used.

**Proposition 1.** Let \( i, j \in R, i \neq j \). Consider \( h_k, k \in R \), satisfying the convex sum property (2). For any matrices \( X \in R^{n \times n} \) and \( P_k \in R^{n \times n}, k \in R \), the following equality holds:

\[
\sum_{k=1}^{r} h_k P_k = \frac{1}{2} \left[ h_i (P_i - P_j + X) + h_j (P_j - P_i + X) \right] + \sum_{k=1}^{r} h_k (2P_k - P_i - P_j + X). \tag{9}
\]

**Proof:** Considering matrices \( P_k, k \in R \), one has that

\[
\sum_{k=1}^{r} h_k P_k = h_i P_i + h_j P_j + \sum_{k=1}^{r} h_k P_k. \tag{10}
\]

From (2), the following property is obtained:

\[
\sum_{k=1}^{r} h_k = 0, \text{ which yields } h_i = -\sum_{k=1}^{r} h_k \text{ for any } i \in R. \tag{11}
\]

Now, using property (11), (10) may be written as

\[
\sum_{k=1}^{r} h_k P_k = -\sum_{k=1}^{r} h_k P_i - \sum_{k=1}^{r} h_k P_j + \sum_{k=1}^{r} h_k P_k
\]

\[
= -h_i P_i - h_i P_j - \sum_{k=1}^{r} h_k P_i - \sum_{k=1}^{r} h_k P_j + \sum_{k=1}^{r} h_k P_k
\]

\[
= -h_i P_i - h_i P_j + \sum_{k=1}^{r} h_k (P_k - P_i - P_j). \tag{12}
\]

Again, considering (11), for any matrix \( X \), (10) can be rewritten as:

\[
\sum_{k=1}^{r} h_k P_k = \sum_{k=1}^{r} h_k (P_k + X)
\]

\[
= (h_i + h_j)X + h_i P_i + h_j P_j + \sum_{k=1}^{r} h_k (P_k + X). \tag{13}
\]

Finally, adding (12) and (13) the result follows. \( \blacksquare \)

Proposition 1 is the key point to establish Theorem 1 which yields new stability conditions for the TS fuzzy system (1). The main result is inspired in the time varying stability results given in Montagner and Peres (2003).

**Theorem 1.** Assume \( |\hat{h}_i| \leq \phi_\rho \) for known positive real numbers \( \phi_\rho \) and \( \rho \in R \). The TS fuzzy system (1) is globally asymptotically stable if for some \( i, j \in R, i \neq j \), there exist matrices \( M_1 \in R^{n \times n} \) and \( M_2 \in R^{n \times n} \), and symmetric matrices \( X \in R^{n \times n} \) and \( P_\rho \in R^{n \times n} \), satisfying the following LMIs:

\[
P_\rho > 0, \quad \forall \rho \in R, \tag{14}
\]

\[
P_\rho - P_i - P_j + X \succeq 0, \quad \forall \rho \in R - \{i, j\}, \tag{15}
\]

\[
\hat{P}_\rho + P_k A_k + A_k^T P_k + P_k A_k + A_k^T P_k \prec 0, \quad k \leq \ell, \tag{16}
\]

where

\[
\hat{P}_\rho = \frac{1}{2} \left[ \pm \phi_\rho (P_i - P_j + X) \pm \phi_\rho (P_j - P_i + X) \right]
\]

\[
+ \sum_{\rho \neq 1}^{\rho \neq 1} \phi_\rho (2P_\rho - P_i - P_j + X)
\]

with the symbol \( \pm \) in the LMI indicating that all possible combinations of + and − must be tested.

**Proof:** Consider a fuzzy Lyapunov function given by:

\[
V(x(t)) = x(t)^T \left( \sum_{k=1}^{r} h_k P_k \right) x(t), \quad V(x(t)) > 0, \quad \forall x(t) \neq 0,
\]

with \( \tilde{P}_\rho \) satisfying (14). The time derivative of \( V(x(t)) \) along the trajectories of (1) is given by (17) (see next page). Now, replacing \( \sum_{k=1}^{r} h_k P_k \) by (9) and considering (2), it yields

\[
\dot{V}(x(t)) = \frac{1}{2} x(t)^T \left( \sum_{\ell=1}^{\rho} \sum_{k=1}^{r} h_k h_k \left[ h_i (P_i - P_j + X) \right] \right) x(t). \tag{18}
\]
\[ \dot{V}(x(t)) = x(t)' \left( \sum_{k=1}^{r} \dot{h}_k P_k \right) x(t) + \dot{x}(t)' \left( \sum_{k=1}^{r} h_k P_k \right) x(t) + x(t)' \left( \sum_{k=1}^{r} h_k A_k \right) x(t) = \ldots \]

Since \(|\dot{h}_k| \leq \phi_0\), by imposing conditions (15) and (16), one can conclude that
\[ \dot{V}(x(t)) \leq \frac{1}{2} x(t)' \left\{ \sum_{k=1}^{r} h_k h_k \left[ \bar{P}_o + \bar{A}_t P_k + \bar{A}_t A_t \right] + \bar{P}_t A_t + \bar{P}_t A_k \right\} x(t) < 0, \] and the result follows.

Repeating Example 1 with Lemma 2 and Theorem 1, the stability regions in terms of parameters \(a\) and \(b\) are showed in Figure 2. Note that Theorem 1 yields a larger stable region than Lemma 2. The stable region marked with \(\times\) in Figure 2 was obtained assuming \(\phi_k = 0.85, \forall k \in \mathcal{R}\). The choice of the indexes \(i\) and \(j\) affects the stability regions found by Theorem 1. In this example, the best stable region of Theorem 1 is obtained with \(i = 4\) and \(j = 1\) (Figure 2), and the worst is obtained with \(i = 3\) and \(j = 4\) (Figure 3). The figures illustrated that the stable region obtained by Theorem 1 is always better than the obtained by Lemma 2.

**Remark 1.** Lemma 2 is a particular case of Theorem 1. In fact, by rewriting
\[ \phi_i (P_i - P_j + X) + \phi_j (P_j - P_i + X) = \phi_i (2P_i - P_i - P_j + X) + \phi_j (2P_j - P_i - P_j + X) \]
and adding \(\sum_{\rho \neq 1 \text{ and } \rho \neq j} \phi_\rho (2P_\rho - P_i - P_j + X)\) to both sides of inequality above, it follows that

![Graph](image-url)

**Fig. 3.** Stability analysis of Lemma 2 (\(\times\)) and Theorem 1 with \(i = 3\) and \(j = 4\) (\(\times\)).

\[ \phi_i (P_i - P_j + X) + \phi_j (P_j - P_i + X) + \sum_{\rho = 1}^{r} \phi_\rho (2P_\rho - P_i - P_j + X) = \sum_{\rho = 1}^{r} \phi_\rho (2P_\rho - M_3), \]

which is equivalent to \(P_\phi\) in Lemma 2. Consequently, the set of LMIs (6)-(8) is a particular case of (14)-(16).

In the next section, a control design method considering Proposition 1 and the addition of slack matrix variables as used in Mozelli et al. (2009a) is proposed. Differing from the results given by Lemma 1, this method allows the control design to be solved using LMIs.

### 5. STATE FEEDBACK DESIGN

For control purposes, consider the parallel distributed compensation (PDC) procedure presented in Tanaka et al. (1998). The feedback TS fuzzy system is given by:
\[ \dot{x}(t) = \sum_{k=1}^{r} h_k h_k (A_k - B_i L_k) x(t), \]
where \(h_k, k \in \mathcal{R}\), satisfies (2) and \(L_k \in \mathbb{R}^{m \times n}\) are the local state feedback gains. The objective now, is to find matrices \(L_k, \forall k \in \mathcal{R}\), such that system (19) is globally asymptotically stable.

The control design is performed using the same ideas used in Mozelli et al. (2009a). From (19), it follows that
\[
\dot{x}(t) - \sum_{k=1}^{r} \sum_{\ell=1}^{r} h_k h_\ell (A_k - B_k L_\ell) x(t) = 0, \quad \forall x(t). \quad (20)
\]

Therefore, given a full rank matrix \( Z \in \mathbb{R}^{n \times n} \) and a scalar \( \mu > 0 \), from null term (20), it follows that
\[
2 \begin{bmatrix} x(t) (Z')^{-1} + \dot{x}(t) \mu(Z')^{-1} \\
\dot{x}(t) - \sum_{k=1}^{r} \sum_{\ell=1}^{r} h_k h_\ell (A_k - B_k L_\ell) x(t) \end{bmatrix} = 0, \quad \forall x(t). \quad (21)
\]

Considering (21), a sufficient condition for stabilization of the system (19) with fuzzy Lyapunov functions is proposed next.

\textbf{Theorem 2.} Assume \( |b_{ij}| \leq \phi_{ij} \) for known positive real numbers \( \phi_{ij} \) and \( \rho \in \mathcal{R} \). The feedback system (19) is stabilizable, with local gains \( L_i = Y_i Z^{-1} \), if for some \( i, j \in \mathcal{R}, i \neq j \) and a scalar \( \mu > 0 \), there exist symmetric matrices \( S, Q_\rho \in \mathbb{R}^{n \times n} \), matrices \( Z \in \mathbb{R}^{n \times n} \), and \( Y_\ell \in \mathbb{R}^{n \times n}, \ell \in \mathcal{R} \), satisfying the following LMIs:
\[
\begin{align*}
Q_\rho &> 0, \quad \forall \rho \in \mathcal{R}, \quad (22) \\
2Q_\rho - Q_i - Q_j + S &> 0, \quad \forall \rho \in \mathcal{R} - \{i, j\}, \quad (23) \\
[Q_\rho - A_\ell Z - Z^\top A_\ell^\top + B_\ell Y_\ell + Y_\ell^\top B_\ell^\top \quad \mu(Z + Z^\top)] &< 0, \quad (24) \\
[Q_\rho - A_\ell Z - Z^\top A_\ell^\top + B_\ell Y_\ell + Y_\ell^\top B_\ell^\top \quad \mu(Z + Z^\top)] &< 0, \quad (25)
\end{align*}
\]

where \( k, \ell = 1, \ldots, r; k \neq \ell \) and
\[
\tilde{Q}_\rho = \frac{1}{2} \left[ \pm \phi_i (Q_i - Q_j + S) \pm \phi_j (Q_j - Q_i + S) + \sum_{\rho \neq i, \rho \neq j} \phi_{ij} (2Q_\rho - Q_i - Q_j + S) \right].
\]

**Proof:** The proof follows the same lines as the proof of Theorem 6 given in [Mozelli et al. (2009a)](https://example.com). Consider a fuzzy Lyapunov function given by:
\[
V(x(t)) = x(t)' \left( \sum_{k=1}^{r} h_k P_k \right) x(t) > 0, \quad x(t) \neq 0.
\]

The time derivative of \( V(x(t)) \) is
\[
\dot{V}(x(t)) = x(t)' \left( \sum_{k=1}^{r} h_k P_k \right) x(t) + 2x(t)' \left( \sum_{k=1}^{r} h_k P_k \right) \dot{x}(t).
\]

Using property (2), the null term (21) and the matrix \( \tilde{P}_\rho \) defined under the conditions of Theorem 1, it follows that
\[
\dot{V}(x(t)) = \dot{x}(t)' \left( \sum_{k=1}^{r} h_k P_k \right) x(t) + 2x(t)' \left( \sum_{k=1}^{r} h_k P_k \right) \dot{x}(t) + 2 \begin{bmatrix} x(t)' (Z')^{-1} + \dot{x}(t) \mu(Z')^{-1} \\
\dot{x}(t) - \sum_{k=1}^{r} \sum_{\ell=1}^{r} h_k h_\ell (A_k - B_k L_\ell) x(t) \end{bmatrix} = 0, \quad \forall x(t). \quad (26)
\]

The middle term on the right-hand side of (26) can be rewritten as (27) (see next page). In fact, premultiplying and postmultiplying (24) and (25) by \( (Z')^{-1} \) and \( (Z^{-1})^{-1} \) respectively, and considering the substitutions
\[
P_k = (Z')^{-1} Q_k Z^{-1}, \quad \forall k, \quad X = (Z')^{-1} S Z^{-1},
\]
\[
\tilde{P}_\rho = (Z')^{-1} \tilde{Q}_\rho Z^{-1} \quad \text{and} \quad L_\rho = Y_i Z^{-1},
\]
one obtains the matrices appearing in (27). Therefore, the scalar function (26) is negative for all \( x(t) \neq 0 \) and by the direct Lyapunov method, system (19) is globally asymptotically stable.

**EXAMPLE 4**

Consider the TS fuzzy system (19) with the following local models:
\[
A_1 = \begin{bmatrix} -3 & -2 \\
8 & 9 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -2 & -3 \\
8 & 19 \end{bmatrix}, \quad A_3 = \begin{bmatrix} -2 & 3 \\
8 & a + 9 \end{bmatrix},
\]
\[
B_1 = \begin{bmatrix} 0 \\
1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\
1 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 \\
b \end{bmatrix}
\]

For \( a \in [1, 6] \) and \( b \in [1, 4] \), this example is solved assuming \( \phi_k = 2, \quad \forall k \in \mathcal{R} \) and \( \mu = 0.4 \). The region where the state feedback controllers can be found with Theorem 2 and Theorem 6 from [Mozelli et al. (2009a)](https://example.com) is exhibited in Figure 4. With the parameters in these intervals, Theorem 2 is able to stabilize the system with a larger region than given by Theorem 6.

6. CONCLUSIONS

In this work, less conservative stability conditions for Takagi-Sugeno fuzzy models were proposed. First, some recent results on stability analysis for TS fuzzy system with fuzzy Lyapunov functions were discussed. After, a method that outperform the existing ones was proposed by applying a relaxation in the method presented in [Tanaka et al. (2003)](https://example.com) and adding a slack variable in a similar way introduced in [Mozelli et al. (2009b)](https://example.com). In addition, a control design is proposed using the simple techniques also presented in [Mozelli et al. (2009a)](https://example.com). The results are obtained via the solution of LMIs, which are efficiently solved by convex programming techniques. Numerical simulations illustrate the efficiency of the results.
\[
\sum_{k=1}^{r} \sum_{\ell=1}^{r} h_k h_{\ell} \left[ \hat{P}_\phi - (Z')^{-1} (A_k - B_k L_\ell) - (A_k - B_k L_\ell) Z^{-1} P_k - \mu (A_k - B_k L_\ell) Z^{-1} + (Z')^{-1} \right] = \\
\sum_{k=1}^{r} h_k \left[ \hat{P}_\phi - (Z')^{-1} (A_k - B_k L_k) - (A_k - B_k L_k) Z^{-1} P_k - \mu (A_k - B_k L_k) Z^{-1} + (Z')^{-1} \right] \\
+ \sum_{k=1}^{r-1} \sum_{\ell=k+1}^{r} h_k h_{\ell} \left[ \hat{P}_\phi - (Z')^{-1} (A_k - B_k L_{\ell}) - (A_k - B_k L_{\ell}) Z^{-1} P_k - \mu (A_k - B_k L_{\ell}) Z^{-1} + (Z')^{-1} \right] \\
\frac{\mu}{\mu [ (Z')^{-1} + Z^{-1} ]} \\
\frac{P_k - \mu (A_k - B_k L_k) Z^{-1} + (Z')^{-1} + \mu [ (Z')^{-1} + Z^{-1} ]}.
\] (27)

Fig. 4. Stabilization region based on Theorem 6 from Mozelli et al. (2009a) (○) and Theorem 2 with \( i = 1, j = 2 \) and \( \mu = 0.4 \) (×).

REFERENCES


