Output feedback stabilization of time-delay switched linear systems

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Abstract: This paper deals with the stabilization issue of switched linear systems with time-delays. Based on previous results related to switched linear systems with no time-delays and on the concept of piecewise quadratic Lyapunov-Krasovskii functionals, we solve the problem of searching output feedback switching laws ensuring stability. Our synthesis method is based on the solution of a set of matrix inequalities associated with the modes of the switched system.

Keywords: Switched systems; Time-delay; Stabilization; Linear matrix inequality.

1. INTRODUCTION

During the last years, the number of publications devoted to the study of time-delay switched systems (TDSS) has considerably grown. Concepts and results from the theory of time-delay systems (see Richard [2003] for a survey and de Oliveira and Geromel [2004] for output feedback control synthesis based on Lyapunov-Krasovskii functionals) have been extended to this class of systems and one of the main topics to attract interest in the researchers community is related to the stability and stabilization problems. Similarly to what can be found in the literature on standard switched systems, the first proposed results in this field were largely based on the use of Common Lyapunov Functions (CLFs), which represent an extension of the standard Common Lyapunov Functions (CLFs) used in the case with no time-delays. For example, Zhai et al. [2003] provided an analysis of stability and $L_2$ gain of linear TDSS with a time-invariant delay under arbitrary switching by means of CLFLs, under some symmetry assumptions on the model. In Xie and Wang [2004], CLFLs were applied to stability analysis and stabilization through the assignment of the input and/or the switching law, while He et al. [2004] proposed a model transformation based on free-weighting matrices to robust stability analysis in the presence of a time-varying delay, based on delay-dependent Lyapunov functionals. The results in Zhai et al. [2003] and Xie and Wang [2004] were partially extended in Sun et al. [2006a], where stability and $L_2$-gain analysis were addressed in the presence of time-varying delays, by introducing delay-dependent stability criteria and formulating exponential stability conditions based on the concept of average dwell-time. In Sun et al. [2006b], further delay-dependent stability criteria were proposed, accounting for time-varying delays and possibly uncertain models by CLFLs by means of a descriptor system approach. Relevant stabilization results can also be found in Kim et al. [2006].

Recently, some authors focused on stability criteria and stabilization methods based on multiple Lyapunov functionals or piecewise quadratic Lyapunov functionals. Sun et al. [2006a] studied stability and $L_2$ gain of such systems through a candidate piecewise quadratic Lyapunov functional and the average dwell-time method in the presence of time-varying delays and both stable and unstable subsystems. A related paper is Wang et al. [2009], where delay-dependent exponential stability criteria have been proposed for interval time-delays and possibly both stable and unstable subsystems, through an average dwell-time method inherited from Sun et al. [2006a]. Sun et al. [2008] applied an LMI approach to the stability analysis in both cases of time-invariant and time-varying delays. In Yan and Özbay [2007], the stability analysis was performed through piecewise quadratic Lyapunov-Razumikhin functions through a delay-dependent/independent minimum dwell-time approach assuming that the time-delays are dependent on the active mode in the system.

In this paper we consider the output feedback stabilization problem for linear TDSS, assuming that the control variable is the switching signal. Our approach exploits piecewise quadratic Lyapunov-Krasovskii functionals and is based on the solution of a set of matrix inequalities associated to the modes of the system. Differently from most of the previous contributions on the topic, our approach allows to consider switched systems in which not all the modes are stable, while preserving the stability of the overall switched system thanks to the choice of the switching law. Moreover, our control strategy also accounts for the stability issue of possible sliding modes, which is usually neglected in the previous literature. Our results are based on a method proposed in Galbusera and Bolzern [2010] as well as on ideas originally proposed in Geromel.
and Colaneri [2006] and Geromel and Deaecto [2009], where the stabilization issue of switched linear systems (with no delays) via switching is studied by exploiting an approach based on Lyapunov-Metzler inequalities. The same concepts are also exploited in Deaecto et al. [2010] in order to address the output feedback stabilization problem for switched linear systems.

The paper is organized as follows. In Section 2 we summarize some definitions and preliminaries; Section 3 is devoted to the analysis of a relevant stabilizing state feedback switching strategy proposed in Galbusera and Bolzern [2010]; Section 4 contains the main contribution of the paper, addressing the output feedback stabilization of switched linear systems with time-delays by means of switching; in Section 5 we present a numerical example and finally some concluding remarks are proposed.

**Notation:** the identity matrix of any dimension is denoted by $I$. For real matrices or vectors, the symbol $(\cdot)^{\top}$ indicates transpose. The squared norm of a signal $\xi(t)$ defined for all $t \geq 0$ is denoted by $\|\xi\|_2^2$. The set of all signals such that $\|\xi\|_2^2 < \infty$ is denoted by $L_2$. For a real matrix $M$, the Hermitian operator $H_e\{\cdot\}$ is defined as $H_e\{M\} = M + M^{\top}$.

**2. DEFINITIONS AND PRELIMINARIES**

Given an index set $\mathcal{P} = \{1, \ldots, N\}$, denote by $\sigma(\cdot)$ the switching signal, which is a piecewise constant, right-continuous mapping from $\mathbb{R}_+\to \mathcal{P}$. Define the following continuous-time switched linear system subject to a single time-invariant delay $h \geq 0$ acting on the state

$$\begin{align*}
\dot{x}(t) &= A_{x(t)}(t) + A_{d(x)}(t-h) \\
y(t) &= C_{\sigma(t)}x(t)
\end{align*}$$

(1)

where $x(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}^q$. Our objective is the synthesis of state and output feedback switching laws (i.e., switching laws based either on $x(t)$ or on $y(t)$) able to ensure that the equilibrium solution $x_{\pi} = 0$ of system (1) is asymptotically stable irrespectively of the value of $h$. The sequel of this section is devoted to introduce some preliminaries to the new contributions presented in this paper.

**2.1 Stabilization using Lyapunov-Metzler inequalities**

Consider the following continuous-time switched linear system with no time-delays

$$\dot{x}(t) = A_{x(t)}x(t)$$

(2)

Referring to such a model, we now introduce some results from Geromel and Colaneri [2006] addressing the stabilization problem via state feedback switching; such results are based on the use of piecewise quadratic Lyapunov functions (see Liberzon [2003] for an introduction) and will be exploited in the sequel of the paper. In particular, consider the following piecewise quadratic Lyapunov function

$$v(x) = \min_{i \in \mathcal{P}} x^{\top} P_i x = \min_{\lambda \in \Lambda} \left( \sum_{i \in \mathcal{P}} \lambda_i x^{\top} P_i x \right)$$

(3)

where $\Lambda = \{\lambda \in \mathbb{R}^N : \sum_{i \in \mathcal{P}} \lambda_i = 1, \lambda_i \geq 0\}$ and $P_i = P_i^{\top} > 0, \forall i \in \mathcal{P}$. The function $v(x)$, in general, is not uniformly differentiable in $x \in \mathbb{R}^n$ as the set $\mathcal{I}(x) = \{i \in \mathcal{P} : v(x) = x^{\top} P_i x\}$ may be composed of more than one element, i.e., the result of minimization (3) is not unique, see Geromel and Colaneri [2006]. Furthermore, introduce the class of Metzler matrices, composed of all square matrices of fixed dimensions with nonnegative off diagonal entries. In particular, we are interested in the class $\mathcal{M}$ of Metzler matrices $\Pi \in \mathbb{R}^{N \times N}$ with the following property, $\forall i \in \mathcal{P}$

$$\pi_{ji} \geq 0, \forall j \in \mathcal{P} \setminus \{i\} \land \sum_{j \in \mathcal{P}} \pi_{ji} = 0$$

(4)

For simplicity, we use the notation $P_{\Pi_i} = \sum_{j \in \mathcal{P}} \pi_{ji} P_j$. A fundamental property of such matrices is the following (see e.g. Geromel and Colaneri [2006], Galbusera [2009]).

**Lemma 1.** For any $\Pi \in \mathcal{M}$ and any $i \in \mathcal{I}(x)$, the inequality $x^{\top} P_{\Pi_i} x \geq 0$ holds.

Indeed, this property follows from (4) and from the fact that whenever $i \in \mathcal{I}(x)$, then $x^{\top} P_{\Pi_i} x \geq x^{\top} P_i x$ for all $j \in \mathcal{P}$. Also consider the following result from Geromel and Colaneri [2006], whose proof exploits Lemma 1.

**Theorem 1.** Assume that there exist matrices $P_i = P_i^{\top} > 0$ and $\Pi \in \mathcal{M}$ such that, $\forall i \in \mathcal{P}$, the following Lyapunov-Metzler inequality holds

$$H_e\{A_i P_i\} + P_{\Pi_i} < 0$$

(5)

Then, the state feedback switching control law

$$\sigma(t) = \arg\min_{i \in \mathcal{P}} v(t)^{\top} P_i v(t)$$

(6)

makes the equilibrium solution $x = 0$ of system (2) globally asymptotically stable.

Observe that, in view of Theorem 1, a necessary condition for the feasibility of condition (5) is $A_i + (\pi_{ii}/2)I$ being Hurwitz, $\forall i \in \mathcal{P}$. This does not imply that the set $\{A_1, \ldots, A_N\}$ must be exclusively composed of Hurwitz matrices, since $\pi_{ii} \leq 0$. On the other side, when all such matrices are Hurwitz the proposed state feedback switching strategy with $\Pi = 0$ preserves stability, since (5) reduces to the standard condition $A_i P_i + P_i A_i < 0$. Finally observe that, thanks to (6), no unstable sliding modes may occur. In fact, switching from any mode $i \in \mathcal{I}(x)$ to $j \in \mathcal{I}(x)$ is possible only if $x^{\top} (A_j P_j + P_j A_j) x \leq x^{\top} (A_i P_i + P_i A_i) x < 0$, thus (3) is strictly decreasing along the system’s trajectories even in the presence of sliding modes, for more details, see Geromel and Colaneri [2006].

Notice that formula (5) is a bilinear matrix inequality (BMI), because of the presence of the Metzler term $\Pi$. Thus, due to non-convexity, finding a feasible solution is not trivial in general. Anyway, a possible simplification procedure is illustrated in Geromel and Colaneri [2006], which reduces the solution of the problem to the combination of a line search and the solution of a set of LMIs. This method adds a further degree of conservatism to the original formulation, but at the same time strongly decreases the computational effort in searching for feasible solutions. The same observation also holds for the criteria that are proposed in the sequel of this paper.

**2.2 A matrix-theoretical result**

In Deaecto et al. [2010], the following matrix-theoretical property was proven, which is reported here since it is
crucial to the development of the subsequent theoretical contributions.

Lemma 2. Given a nonsingular matrix $V$ and symmetric matrices $Y_i$ and $X_i$, $\forall i \in \mathcal{P}$, of compatible dimensions satisfying
\[
\begin{bmatrix}
Y & I \\
I & X_i
\end{bmatrix} > 0
\]  
(7)

it is possible to determine nonsingular matrices $U_i$ and symmetric matrices $\hat{Y}_i$ and $\hat{X}_i$ such that, $\forall i \in \mathcal{P}$,
\[
\hat{S}_i^{-1} = \begin{bmatrix}
Y & I \\
I & X_i
\end{bmatrix} > 0, \quad \hat{S}_i = \begin{bmatrix}
X_i & \ast \\
U_i' & \hat{X}_i
\end{bmatrix} > 0
\]  
(8)

The most interesting consequence of Lemma 2 is the existence of the nonsingular constant matrix
\[
\hat{\Gamma} = \begin{bmatrix}
Y & I \\
V' & 0
\end{bmatrix}
\]  
(9)
such that
\[
\hat{\Gamma}' \hat{S}_i \hat{\Gamma} = \begin{bmatrix}
Y & I \\
I & X_i
\end{bmatrix} > 0
\]  
(10)
for all $i \in \mathcal{P}$. Notice that, in the above partitioned matrices, all blocks are square with the same dimensions.

3. STATE FEEDBACK STABILIZATION

This section is devoted to the introduction of a criterion for the stabilization of the switched time-delay system (1) by means of the synthesis of a suitable switching control law. A basic assumption is that the state of the system is available, so that a state feedback control law can be synthesized. As shown in the sequel, in this approach the synthesis of stabilizing switching laws is done by finding, for system (1), a piecewise quadratic Lyapunov-Krasovskii functional.

Theorem 2. Consider system (1) and assume that there exist matrices $P_i = P_i' > 0$, $Q = Q' > 0$ and $\Pi \in \mathcal{M}$ such that, $\forall i \in \mathcal{P}$,
\[
\begin{bmatrix}
H_{\xi} \{ A_j'P_j \} + Q + P_{P_i} \ast & A_{d_i}P_i \\
A_{d_i}P_i & -Q
\end{bmatrix} < 0
\]  
(11)

Then the switching law
\[
\sigma(x(t)) = \arg \min_{i \in \mathcal{P}} x(t)'P_ix(t)
\]  
(12)

makes the equilibrium solution $x_t = 0$ of system (1) asymptotically stable, irrespectively of the value of the time-delay $h$.

The proof of Theorem 2 is reported in Galbusera and Bolzern [2010] and is based on the introduction of the following Lyapunov-Krasovskii functional associated to mode $i \in \mathcal{P}$
\[
v_i(x_t) = x(t)'P_ix(t) + \int_{t-h}^{t} x(\theta)'Qx(\theta)d\theta
\]  
(13)

and the corresponding piecewise quadratic Lyapunov-Krasovskii functional
\[
v(x_t) = \min_{i \in \mathcal{P}} v_i(x_t)
\]  
(14)
together with the switching law $\sigma(x_t) = \arg \min_{i \in \mathcal{P}} v_i(x_t)$. Observe that, since the integral term in (14) does not depend on the mode $i \in \mathcal{P}$, such a switching law is equivalent to (12). Therefore, at time $t$, the switching law proposed in (12) does only depend on the state $x(t)$ and not on past values.

Since, in general, $v(x_t)$ is not uniformly differentiable over its domain, in the above reference the concept of upper Dini derivative is exploited for the manipulation of such a functional. In the sequel of this section, we detail the analysis of some properties of the control solution proposed in Theorem 2.

Remark 1. We detail the stability analysis in correspondence of possible sliding modes occurring in the time evolution of system (1) subject to the switching law (12). In view of the properties of (12), the existence of a sliding mode along a switching surface $\Omega$ at the boundary of activation regions of modes $i \in \mathcal{I}(x(t))$ requires that the following inequalities, (see Liberzon [2003])
\[
\hat{v}_i|_{\sigma(t)=\nu} \geq \hat{v}_i|_{\sigma(t)=\sigma}, \quad \forall i \in \mathcal{I}(x(t))
\]  
(15)

hold. For $k, \nu \in \mathcal{P}$, $\hat{v}_k|_{\sigma(t)=\nu}$ denotes the time derivative of $v_k$ along the trajectories of system (1) at time $t$, assuming that $\sigma(t) = \nu$. In our case, the conditions in (15) can be equivalently rewritten as follows
\[
x(t)' (H_{\xi} \{ A_j' (P_j - P_i) \}) x(t) + 2x(t)' (P_j - P_i) A_{d_i} x(t-h) \leq 0
\]  
(16)

The switching signal is not uniquely defined on the sliding surface $\Omega$. Therefore, we can assume $\sigma(t) = i$ without loss of generality and prove that $v_i(x_t)$ defined in (13) uniformly decreases along the corresponding Filippov solution of the system. In fact, denoting $A_{a_i} = \sum_{j \in \mathcal{I}(x(t))} \alpha_j A_j$, and $A_{d_i} = \sum_{j \in \mathcal{I}(x(t))} \alpha_j A_{d_j}$, for all $\alpha_j \geq 0$ such that $\sum_{j \in \mathcal{I}(x(t))} \alpha_j = 1$, we obtain
\[
\hat{v}_i(x_t) = x(t)' (H_{\xi} \{ A_j' P_j \}) x(t) + 2x(t)' P_j A_{d_i} x(t-h) + x(t)' Qx(t) - x(t-h)' Qx(t-h) \leq \sum_{j \in \mathcal{I}(x(t))} \alpha_j \xi(t)' \begin{bmatrix}
H_{\xi} \{ A_j' P_j \} + Q & \ast \\
\ast & -Q
\end{bmatrix} \xi(t) < 0
\]  
(17)

where $\xi(t) = [x(t)', x(t-h)']$ the inequalities follow from (16), the negative definiteness of the Lyapunov-Krasovskii inequalities (11) and the fact that both indices $(i, j)$ are elements of $\mathcal{I}(x(t))$. Therefore any sliding mode of system (1) subject to the switching law (12) is asymptotically stable.

Assume $h = 0$. In this particular case, the result of Theorem 1 can be applied and the switching strategy (6) is stabilizing if
\[
H_{\xi} \{ \hat{A}_i P_i \} + P_{P_i} < 0
\]  
(17)
with $\hat{A}_i = A_i + A_{d_i}$. This observation emphasizes the role of the term $P_{P_i} = \sum_{j \in \mathcal{P}} \pi_{i,j} P_j$ in relaxing the stability requirement on the single subsystems. In fact, observe that the asymptotic stability of $\hat{A}_i$, $\forall i \in \mathcal{P}$, is not necessary for the asymptotic stability of the switched system (1). Indeed, a necessary condition for asymptotic stability under the switching law (12) can be derived as being $\hat{A}_i + (\pi_{i,i}/2) I$ Hurwitz.
4. OUTPUT FEEDBACK STABILIZATION

In this section, we introduce the main contribution of this paper, addressing the problem of designing an output feedback switching controller able to stabilize system (1), only based on the available information represented by the output signal $y(t)$. This can be done by suitably extending the theoretical results presented in the previous section. To this end, we introduce the filter

$$\hat{x}(t) = \hat{A}_i(t)\hat{x}(t) + \hat{B}_i(t)y(t)$$

with $\hat{x}(t) \in \mathbb{R}^n$. Observe that (18) only depends on the output $y(t)$ of system (1). Defining $\hat{x}(t) = [x(t)\; \hat{x}(t)]'$, we can combine together models (1) and (18), obtaining the following augmented system

$$\hat{x}(t) = \hat{A}_i(t)\hat{x}(t) + \hat{A}_{\text{det}}(t)\hat{x}(t-h)$$

where the state space matrices are given by

$$\hat{A}_i = \begin{bmatrix} A_i & 0 \\ B_iC_i & A_i \end{bmatrix}, \quad \hat{A}_{\text{det}} = \begin{bmatrix} A_{\text{det}} & 0 \\ 0 & 0 \end{bmatrix}$$

As a consequence, in this framework, we aim at finding a switching law $\sigma(\hat{x}(t))$ such that the closed-loop system (19) fulfills the assumptions of Theorem 2 with matrices $A_i$, $A_{\text{det}}$, $P_i$ and $Q$ replaced by $\hat{A}_i$, $\hat{A}_{\text{det}}$, $\hat{P}_i$ and $\hat{Q}$, respectively, $\forall i \in \mathcal{P}$, where $\hat{P}_i = \hat{P}_i > 0$, $\hat{Q} = \hat{Q}' \geq 0$ are assumed to have the following structures

$$\hat{P}_i = \begin{bmatrix} Y & V \\ V' & \hat{Y}_i \end{bmatrix}, \quad \det(V) \neq 0$$

$$\hat{Q} = \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}, \quad Q > 0$$

In view of Lemma 2, the latter expression and the non-singularity of matrix $\hat{Q}$ makes the equilibrium $x_i = 0$ of system (1) asymptotically stable, irrespectively of the value of the time-delay $h$.

**Proof.** First observe that stability of the null equilibrium for system (19) implies stability of the null equilibrium for system (1). Now consider system (19) and define the following Lyapunov-Krasovskii functional associated to mode $i \in \mathcal{P}$

$$v_i(\hat{x}_i) = \hat{x}(t)'\hat{P}_i\hat{x}(t) + \int_{t-h}^{t} \hat{x}(\theta)'\hat{Q}\hat{x}(\theta)d\theta$$

where $\hat{P}_i$ and $\hat{Q}$ are specified as in (20), (21). Correspondingly, introduce the following piecewise quadratic Lyapunov-Krasovskii functional for the switched system (19)

$$v(\hat{x}_i) = \min_{\forall i \in \mathcal{P}} v_i(\hat{x}_i)$$

Hence, a stabilizing switching law can be found for system (19) by applying Theorem 2, thus leading to the conditions

$$\begin{bmatrix} H_c \{\hat{A}_i\hat{P}_i\} + \hat{Q} + \hat{P}_i & * \\ * & -\hat{Q} \end{bmatrix} < 0$$

where $\hat{P}_i = \sum_{j \in \mathcal{P}} \pi_{ij}\hat{P}_j$. The associated switching law is

$$\sigma(\hat{x}(t)) = \arg \min_{\forall i \in \mathcal{P}} v_i(\hat{x}_i) = \arg \min_{\forall i \in \mathcal{P}} \hat{x}(t)'\hat{Y}_i\hat{x}(t)$$

where the last equality depends on the structure of $\hat{P}_i$. Observe that (29) can not be satisfied with the choice of $\hat{Q} \geq 0$ given in (21). In fact, we will determine a solution located at the boundary since as it will be clear in the sequel, the null element of $\hat{Q}$ in the main diagonal of (29) is situated in a null row and column. Now assume that inequality (23) holds. Then, the following condition also holds, $\forall i \in \mathcal{P}$

$$\begin{bmatrix} Z_i & * \\ - Z_i & Y \end{bmatrix} \geq 0$$

Since $Y > 0$ and $Z_i > 0$, $\forall i \in \mathcal{P}$, applying the Schur complement formula to the latter expression we obtain the equivalent condition

$$Y > Z_i, \forall i \in \mathcal{P}$$

As a consequence, defining $X_i = Z_i^{-1}$, $\forall i \in \mathcal{P}$ and exploiting again the Schur complement formula, the following inequality is obtained

$$\begin{bmatrix} Y & * \\ I & X_i \end{bmatrix} \geq 0$$

In view of Lemma 2, the latter expression and the non-singularity of matrix $V$ imply that there exist nonsingular matrices $U_i$ and symmetric matrices $\hat{Y}_i$ and $\hat{X}_i$ such that, $\forall i \in \mathcal{P}$,

$$\hat{S}_i^{-1} = \begin{bmatrix} Y & * \\ V' & \hat{Y}_i \end{bmatrix} > 0, \quad \hat{S}_i = \begin{bmatrix} X_i & * \\ U_i' & \hat{X}_i \end{bmatrix} > 0$$

Let $\hat{P}_i = \hat{S}_i^{-1}$, as in (20). Furthermore, consider the nonsingular matrix $\hat{\Gamma}$ as in (9) and observe that, $\forall i \in \mathcal{P}$, (10) holds. We can now proceed by multiplying the inequality (29) to the right by $\text{diag}(\hat{S}_i, \hat{\Gamma}, I)$ and to the left

$$\hat{A}_i = V^{-1}(N_i - YA_i - L_iC_i)(Z_i - Y)^{-1}V$$

$$\hat{B}_i = V^{-1}L_i$$

the switching law

$$\sigma(\hat{x}(t)) = \arg \min_{\forall i \in \mathcal{P}} \hat{x}(t)'V'(Y - Z_i)^{-1}V\hat{x}(t)$$

which implies that the null equilibrium $x_i = 0$ of system (1) is asymptotically stable, irrespectively of the value of the time-delay $h$. Therefore, a stabilizing switching law can be found for system (19) by applying Theorem 2, thus leading to the conditions

$$\begin{bmatrix} H_c \{\hat{A}_i\hat{P}_i\} + \hat{Q} + \hat{P}_i & * \\ * & -\hat{Q} \end{bmatrix} < 0$$

where $\hat{P}_i = \sum_{j \in \mathcal{P}} \pi_{ij}\hat{P}_j$. The associated switching law is

$$\sigma(\hat{x}(t)) = \arg \min_{\forall i \in \mathcal{P}} v_i(\hat{x}_i) = \arg \min_{\forall i \in \mathcal{P}} \hat{x}(t)'\hat{Y}_i\hat{x}(t)$$

where the last equality depends on the structure of $\hat{P}_i$. Observe that (29) can not be satisfied with the choice of $\hat{Q} \geq 0$ given in (21). In fact, we will determine a solution located at the boundary since as it will be clear in the sequel, the null element of $\hat{Q}$ in the main diagonal of (29) is situated in a null row and column. Now assume that inequality (23) holds. Then, the following condition also holds, $\forall i \in \mathcal{P}$

$$\begin{bmatrix} Z_i & * \\ - Z_i & Y \end{bmatrix} \geq 0$$

Since $Y > 0$ and $Z_i > 0$, $\forall i \in \mathcal{P}$, applying the Schur complement formula to the latter expression we obtain the equivalent condition

$$Y > Z_i, \forall i \in \mathcal{P}$$

As a consequence, defining $X_i = Z_i^{-1}$, $\forall i \in \mathcal{P}$ and exploiting again the Schur complement formula, the following inequality is obtained

$$\begin{bmatrix} Y & * \\ I & X_i \end{bmatrix} \geq 0$$

In view of Lemma 2, the latter expression and the non-singularity of matrix $V$ imply that there exist nonsingular matrices $U_i$ and symmetric matrices $\hat{Y}_i$ and $\hat{X}_i$ such that, $\forall i \in \mathcal{P}$,

$$\hat{S}_i^{-1} = \begin{bmatrix} Y & * \\ V' & \hat{Y}_i \end{bmatrix} > 0, \quad \hat{S}_i = \begin{bmatrix} X_i & * \\ U_i' & \hat{X}_i \end{bmatrix} > 0$$

Let $\hat{P}_i = \hat{S}_i^{-1}$, as in (20). Furthermore, consider the nonsingular matrix $\hat{\Gamma}$ as in (9) and observe that, $\forall i \in \mathcal{P}$, (10) holds. We can now proceed by multiplying the inequality (29) to the right by $\text{diag}(\hat{S}_i, \hat{\Gamma}, I)$ and to the left
by its transpose, thus obtaining the following equivalent sufficient condition for the stabilization of system (19)

$$
H_e \left\{ \hat{\Gamma}' S_i \hat{A}_i \hat{\Gamma} + \hat{\Gamma}' \tilde{S}_i \tilde{Q} \tilde{S}_i \hat{\Gamma} + \hat{\Gamma}' \tilde{S}_i \tilde{P} \hat{P}_i \tilde{S}_i \hat{\Gamma} \right\} < 0
$$

Exploiting the definitions (21), (25) and the relationship $U_i = (I - X_i Y)(V Y)^{-1}$ obtained from Lemma 2, we get

$$
\hat{\Gamma}' S_i \hat{A}_i \hat{\Gamma} = \left[ \begin{array}{cc} A' Y + C' L_i' & A'_i' \\ L'_i & X_i A'_i \end{array} \right] (36)
$$

$$
\hat{\Gamma}' \tilde{S}_i \tilde{Q} \tilde{S}_i \hat{\Gamma} = \left[ \begin{array}{c} I \\ X_i \end{array} \right] Q [I \ X_i] (37)
$$

$$
\hat{\Gamma}' \tilde{S}_i \tilde{P} \tilde{S}_i \hat{\Gamma} = \left[ \begin{array}{cc} A'_i \ Y \ A'_{ii} \\ 0 & 0 \end{array} \right] (38)
$$

where $M_i = (Y A_i + L_i C_i) X_i + V \tilde{A} U_i'$. Moreover, as shown in Deaecto et al. [2010], thanks to property (10) and defining $\tilde{S}_{ij} = \tilde{S}_i - \tilde{S}_j$, the following equalities hold, $\forall i \neq j \in \mathcal{P}$,

$$
\hat{\Gamma}' (\tilde{S}_i \tilde{S}_j - \tilde{S}_i \tilde{S}_j) \hat{\Gamma} = \hat{\Gamma}' (\tilde{S}_j \tilde{S}_i \tilde{S}_j - \tilde{S}_i \tilde{S}_j \tilde{S}_i) \hat{\Gamma}
$$

$$
= \hat{\Gamma}' \tilde{S}_j \tilde{S}_i \hat{\Gamma} + \hat{\Gamma}' \tilde{S}_i \tilde{S}_j \hat{\Gamma}(\hat{\Gamma}' \tilde{S}_i \tilde{S}_j)^{-1} \hat{\Gamma}' \tilde{S}_j \tilde{S}_i \hat{\Gamma}
$$

$$
= \begin{bmatrix} 0 & * \\ 0 & \Xi_{ij} \end{bmatrix}
$$

with $\Xi_{ij} = (X_i - X_j) + (X_i - X_j)(X_j - Y^{-1})^{-1} (X_i - X_j)$, where the calculation of $(\hat{\Gamma}' \tilde{S}_i \tilde{S}_j)^{-1}$ in the third equality exploits the matrix inverse lemma. Note that $X_i - Y^{-1}$ is invertible in view of (32). On the other hand, remembering the definition $X_i = Z_i^{-1}$, we can multiply inequality (23) to the right and to the left by $\text{diag} \{ X_i, X_j, I \}, \forall i \in \mathcal{P}$, and calculate the Schur complement of the obtained expression with respect to the last two rows and columns, which can be written as follows, $\forall i \neq j \in \mathcal{P}$,

$$
X_i R_{ij} X_j > Y^{-1} - X_i + (X_i - Y^{-1})(X_j - Y^{-1})^{-1}(X_i - Y^{-1})
$$

$$
= \Xi_{ij}
$$

where the last equality is obtained by writing $X_i - Y^{-1} = (X_i - X_j) + (X_j - Y^{-1})$ and performing the indicated products. Consequently, taking into account that $\Pi \in \mathcal{M}$, we have

$$
\hat{\Gamma}' \tilde{S}_i \tilde{P} \hat{P}_i \tilde{S}_i \hat{\Gamma} = \hat{\Gamma}' \tilde{S}_i \left( \sum_{i \in \mathcal{P}} \pi_{ji} \tilde{S}_j \right) \tilde{S}_i \hat{\Gamma}
$$

$$
= \hat{\Gamma}' \tilde{S}_i \left( \sum_{i \in \mathcal{P} \setminus \{i\}} \pi_{ji} (\tilde{S}_j^{-1} - \tilde{S}_i^{-1}) \right) \tilde{S}_i \hat{\Gamma}
$$

$$
= \sum_{i \in \mathcal{P} \setminus \{i\}} \pi_{ji} \hat{\Gamma}' (\tilde{S}_i^{-1} - \tilde{S}_i^{-1}) \tilde{S}_j \hat{\Gamma}
$$

$$
\leq \begin{bmatrix} 0 & * \\ 0 & \Xi_{ij} \end{bmatrix}
$$

Exploiting formulas (36), (37), (38) and (39), it results that condition (35), where the last null row and column have been eliminated, is implied by the following one, for all $i \in \mathcal{P}$

$$
\begin{bmatrix}
H_e \left\{ A'_i Y + C'_i L'_i \right\} A_i + M'_i

A'_i Y + H_e \left\{ X_i A_i' \right\} + X_i P_{R_i} X_i

\end{bmatrix} < 0
$$

subject to (23). Observe that, performing the Schur complement with respect to the last term, even ignoring the Metzler term $P_{R_i}$, the obtained inequality is still nonlinear due to the presence of both the matrix variable $Q$ and its inverse. Anyway, let us define the new variables $N_i = M_i X_i^{-1}$, $\forall i \in \mathcal{P}$. Then, multiplying the resulting inequality to the right and to the left by $\text{diag} \{ I, Z_i, I, Q \}$, we obtain condition (22), which together with (23) ensures that system (19) subject to the switching law (26) is asymptotically stable irrespectively of the value of the time-delay $h$. The expression in (26) is obtained from (30) thanks to (34) and the relationship $\tilde{S}_{ij}^{-1} = P_i$, which imply

$$
\hat{Y}_i = V' \hat{Y}_i' (Y - X_i^{-1})^{-1} V = V' \hat{Y}_i' (Y - Z_i)^{-1} V
$$

Finally, the expression in (24) for $\hat{A}_i$ is immediately derived from the definitions of $L_i$ and $N_i$. Thus, the proof is complete.

Remark 2. Matrices $\hat{Y}_i$, $\forall i \in \mathcal{P}$, calculated as in (41), can be used in order to compute the switching surfaces of system (19). In fact, a switching surface $\Omega_{i,j}$ at the boundary of the activation regions of modes $i \neq j \in \mathcal{P}$ can be characterized by solving the equality $v_i = v_j$, or equivalently $\hat{X}_i \hat{Y}_i \hat{x} = \hat{X}_j \hat{Y}_j \hat{x}$, in view of (26).

Using the same rationale as before, it is very easy to show that a necessary condition for (19) being asymptotically stable is that, $\forall i \in \mathcal{P}$, $\hat{A}_i + \hat{A}_{ii} + (\pi_{ii}/2)I$ is Hurwitz. Now notice that such matrix is lower block triangular by definition. Then, also in this case a necessary condition for the delay-independent stabilization of the switched system is that all matrices $\hat{A}_i + \hat{A}_{ii} + (\pi_{ii}/2)I$ are Hurwitz.

5. NUMERICAL EXAMPLE

Consider system (1) with $\mathcal{P} = \{1, 2\}$ and parameters

$$
A_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad A_{d1} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0 \end{bmatrix}, \quad C_1 = [0 \ 1]
$$

$$
A_2 = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \quad A_{d2} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad C_2 = [1 \ 0]
$$

Observe that the classical necessary condition for each subsystem $i \in \mathcal{P}$ being asymptotically stable irrespectively of $h \geq 0$, i.e., $\hat{A}_i = A_i + A_{d_i}$ being Hurwitz, is not fulfilled by any subsystem. In fact, the associated eigenvalues are $\lambda_{1,2}(\hat{A}_1) \approx 0.05 \pm j$ and $\lambda_{1,2}(\hat{A}_2) = \{0.1, -0.9\}$. Applying the criterion presented in Theorem 3, a feasible solution was found to the switched output feedback stabilization problem for $\pi_{21} = \pi_{12} = 1.5$. Correspondingly, fixing $V = \text{diag} \{1, 1\}$, the parameters defining (18) are

$$
\hat{A}_1 = \begin{bmatrix} -0.8768 & 1.1677 \\ -2.9442 & 0.3457 \end{bmatrix}, \quad \hat{B}_1 = \begin{bmatrix} 2.8438 \\ -2.7649 \end{bmatrix}
$$

$$
\hat{A}_2 = \begin{bmatrix} -1.0014 & 0.5085 \\ -0.3879 & -0.7463 \end{bmatrix}, \quad \hat{B}_2 = \begin{bmatrix} -2.5742 \\ -2.0329 \end{bmatrix}
$$
while the switching surfaces are defined by \( \dot{x}_2 \approx 2.8241 \dot{x}_1 \) and \( \dot{x}_2 \approx -0.0583 \dot{x}_1 \). Therefore, Theorem 3 ensures that, for any time-delay \( h \geq 0 \), the system is stabilized by applying the switching law (26). In the simulation results reported next, we set \( h = 10 \) and \( x_0 = [4 \ 4]' \), \( \dot{x}(0) = [0 \ 0]' \). Figures 1 and 2 represent the state trajectories of the switched system, with \( x(t) = [x^{(1)}(t) \ x^{(2)}(t)]' \), while Figure 3 represents the value of the associated Lyapunov-Krasovskii functional along time.

![Figure 1: State \( x^{(1)}(t) \) (solid/dashed line: mode 1/2).](image1)

![Figure 2: State \( x^{(2)}(t) \) (solid/dashed line: mode 1/2).](image2)

![Figure 3: The Lyapunov-Krasovskii functional.](image3)

6. CONCLUSIONS

In this paper we addressed the output feedback stabilization problem for switched linear systems subject to a time delay acting on the state. Our criterion is based on the use of piecewise quadratic Lyapunov-Krasovskii functionals and requires the solution of a set of matrix inequalities in order to find a stabilizing switching law. In our approach, it is not necessary that all modes in the system are stable in order to find a suitable solution. Moreover, our formulation also accounts for the stability of possible sliding modes.

REFERENCES


